David vs Goliath (You against the Markets),

A Dynamic Programming Approach to Separate the Impact and Timing of Trading Costs

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Contents

1 Abstract

We develop a fundamentally different stochastic dynamic programming model of trading costs. Built on a strong theoretical foundation, our model provides insights to market participants by splitting the overall move of the security price during the duration of an order into the Market Impact (price move caused by their actions) and Market Timing (price move caused by everyone else) components. We derive formulations of this model under different laws of motion of the security prices, starting with a simple benchmark scenario and extending this to include multiple sources of uncertainty, liquidity constraints due to volume curve shifts and relate trading costs to the spread.

We develop a numerical framework that can be used to obtain optimal executions under any law of motion of prices and demonstrate the tremendous practical applicability of our theoretical methodology including the powerful numerical techniques to implement them. Our decomposition of trading costs into Market Impact and Market Timing allows us to deduce the zero sum game nature of trading costs. It holds numerous lessons for dealing with complex systems, wherein reducing the complexity by splitting the many sources of uncertainty can lead to better insights in the decision process.

2 Introduction

The recent blockbuster book, David and Goliath: Underdogs, Misfits, and the Art of Battling Giants (Gladwell 2013), talks about the advantages of disadvantages, which in the legendary battle refers to (among other things) the nimbleness that David possesses due to his smaller size and lack of armor, that comes in handy while defeating the massive and seemingly unbeatable Goliath. Despite the inspiring tone of the story the efforts of the most valiant financial market participant can seem puny and turn out to be inadequate, as it gets undone when dealing with the gargantuan and mysterious temperament of uncertainty in the markets. A trader's conundrum is whether (and how much) to trade during a given interval or wait for the next interval when the price momentum is more favorable to his direction of trading. We aim to provide mechanisms that can aid participants and make their life easier when confronting the markets; but given the nature of uncertainty in the social sciences, any weapon will prove to be insufficient compared to the sling shot that delivered the fatal blow to Goliath, until perhaps, one can discern the ability to read the minds of all the market participants.

We develop a fundamentally different stochastic dynamic programming model of trading costs (section 4) based on the Bellman principle of optimality. Built on a strong theoretical foundation, this model can provide insights to market participants by splitting the overall move of the security price during the duration of an order into the Market Impact (price move caused by their actions) and Market Timing (price move caused by everyone else) components. Plugging different distributions of prices and volumes into this framework can help traders decide when to bear higher Market Impact by trading more in the hope of offsetting the cost of trading at a higher price later. We derive formulations of this model under different laws of motion of the security prices. We start with a benchmark scenario and extend this to include multiple sources of uncertainty, liquidity constraints due to volume curve shifts and relate trading costs to the spread (section 6).

The unique aspect of our approach to trading costs is a method of splitting the overall move of the security price during the duration of an order into two components (Collins & Fabozzi 1991; Treynor 1994; Yegerman & Gillula 2014). One component gives the costs of trading, that arise from the decision process that went into executing that particular order, as captured by the price moves caused by the executions that comprise that order. The other component gives the costs of trading, that arise due to the decision process of all the other market participants, during the time this particular order was being filled. This second component is inferred, since it is not possible to calculate it directly (at least with the present state of technology and publicly available data) and it is the difference between the overall trading costs and the first component, which is the trading cost of the executions that make up that order alone. The first and the second component arise due to competing forces, one from the actions of a particular participant, and the other from the actions of everyone else, that would be looking to fulfill similar objectives.

We develop a numerical technique (section 5) that can be used to obtain optimal executions under any law of motion of prices, using a modification of the technique for pricing American options (Longstaff & Schwartz 2001). Our results demonstrate the tremendous practical applicability of our theoretical framework including the numerical techniques to implement them.

The decomposition of trading costs into Market Impact and Market Timing allows us to deduce the zero sum game nature of trading costs (section 3.7). It holds numerous lessons for dealing with complex systems, wherein reducing the complexity by splitting the many sources of uncertainty can lead to better insights in the decision process[1].

2.1 Deeper Intuition from Realistic Trading Situations

Naturally, it follows that each particular participant can only influence to a greater degree the cost that arises from his actions as compared to the actions of others over which he has lesser influence, but an understanding of the second component can help him plan and alter his actions to counter any adversity that might arise from the latter. Any good trader would do this intuitively as an optimization process, that would minimize costs over two variables direct impact and timing, the output of which recommends either slowing

[1] ***To elaborate on this, in any social system it would be helpful to first distinguish the different participants and how their actions contributes to uncertainty. If this is possible, then understanding these components of uncertainty can sometimes help in the analysis of social systems. For example, if we are looking to analyze the shopping patterns in a mall, if we can distinguish shoppers who buy on impulse and shoppers who buy after looking for discounts, we might be better able to forecast sales and analyze this system better. Also, our study can aid in the understanding of complex non-linear phenomena, such as the evolution of prices in financial markets by considering the price changes as being caused by multiple sources of uncertainty. Such an approach of understanding the various sources of uncertainty can be useful in the study of complicated physical phenomena as well.***

down or speeding up his executions. With our methodology, traders now actually have a quantitative indicator to fine tune their decision process. When we decompose the costs, it would be helpful to try and understand how the two sub costs could vary as a proportion of the total. The volatility in these two components, which would arise from different sources (market conditions) would require different responses and hence would affect the optimization problem mentioned above, invoking different sorts of handling. Hence, based on an understanding of two components and the situation at hand, traders would know which cost would be the more unpredictable one and hence focus their efforts on minimizing the costs arising from that component.

The key innovation can be explained as follows:

1. A jump up in price on an execution that comprises a buy order is considered adverse and attributed as impact, while a fall in price is not. Yes, the price could fall further if not for the backstop provided by the executions that comprise the buy order; but the key aspect to remember here is the bilateral nature of trading. A price fall for the buyer (or a benefit for him) is impact for the seller (and hence adverse); and the seller bears the impact cost in this case. To understand this better, we need to remember that if there is a lack of liquidity a buyer can only bid up the price in the hopes of obtaining enough shares to meet his demand and it is these jumps in price in a direction, adverse to his direction of trading that are attributed as his market impact.

2. Most trading cost models consider elaborate theories of the price drifting around, but what actually happens during the transfer of securities is one party, usually, has an upper hand and that is the portion we look to measure as impact for the other party. The key fallout from measuring impact this way is that we have a better way to measure the effect of our actions from when we have a concrete advantage, to when we are okay to put up with a certain disadvantage.

3. The message from this reality is that despite our ambitions to optimize the entire trading process, what we can control is the market impact due to our trades; the market timing, which is the impact for our counter parties is dependent on the decision process of these other market participants and hence beyond the domain of what we can hope (or choose) to optimize.

4. While no measure of trading costs is perfect and complete, this methodology goes a long way in actually providing tangible ways for someone to understand the effect of their decision process and the associated implementation of trades.

Another analogy to understand this methodology is to think of each execution as effecting a state transition from one price level to another. The impact is then the cost or charge involved to make the state transition. We can also think of the change in price levels as moving from one station to another in a train and the ticket price is the cost involved to make this journey. If there is excess demand to travel from one station to another, the ticket price, which is the same for everyone at a particular point in time, changes accordingly and only those that are willing to pay can make the journey. That we are considering the state transitions for each execution at millisecond intervals means that we are building from the bottom up and aggregating smaller effects into an overall impact number for the order based on the executions that comprise it. Theoretically since it is possible that multiple parties could execute simultaneously (two or more buyers and / or sellers on each side), the question of which of the parties is more responsible for causing the price level to change and whether there needs to be a proportional allocation of the price jump does not set in, since all the parties are travelers on the same journey and they all have to pay the ticket price. Though, for executions that happen through a continuous auction process at larger intervals of time, a proportional allocation based on the size of each parties execution might be a possible alternative and will be pursued in later papers.

Figures 1 and 2 show the reversion in the price after an order has completed, broken down by volatility and momentum buckets. The full order sample includes 148,812 orders from 70+ countries with 17 countries having at-least a thousand orders each. The reversion is based on two measures:

1. In time, 5 minutes and 60 minutes after an order has completed.
2. In multiples of the order size, one times and five times the size of the order.

The five Trade Momentum buckets are based on the side adjusted percentage return during the order's trading interval:

1. Significant Adverse (<-2%)
2. Adverse (-1/3% thru -2%)
3. Neutral (-1/3% thru +1/3%)
4. Favorable (+1/3% thru 2%)
5. Significant Favorable (>+2%)

The four Trade Volatility buckets are based on the coefficient of variation of prices during the execution horizon:

1. High Volatility (>0.0050)
2. Moderate Volatility (0.0010 thru 0.0050)
3. Low Volatility (0.000000000000001 thru 0.0010)
4. No Volatility (<= 0.000000000000001)

The size of the bubbles indicates the relative magnitude of the order and its position on the vertical axis signifies the reversion amount in basis points. The box and and the whisker capture the areas where 25% and 75% of the sample resides. Not surprisingly, the momentum is higher in periods of greater volatility, as seen more clearly from the measures based on multiples of the order size (right half of the figures 1 and 2). The higher volatility accentuates the efforts required to trade in such an environment. This illustrates the issue that traders face and the optimization process that is followed where they try to benefit from positive momentum and try to avoid adverse momentum by trading more when adverse momentum is anticipated, while being conscious of the level of volatility.

2.2 Related Literature

Building on the foundation laid by (Bertsimas & Lo 1998), another popular way to decompose trading costs is into temporary and permanent impact (See Almgren & Chriss 2001; Almgren 2003; and Almgren, Thum, Hauptmann & Li 2005). While the theory behind this approach is extremely elegant and considers both linear and nonlinear functions of the variables for estimating the impact, a practical way to compute it requires measuring the price a certain interval after the order. This interval is ambiguous and could lead to lower accuracy while using this measure.

More recent extensions include: minimizing the mean and variance of the costs of trading for the case of market orders only to derive explicit formulas for the optimal trading strategies (Huberman & Stanzl 2005); considering quadratic variation as a reasonable risk measure rather than variance, (Forsyth, Kennedy, Tse & Windcliff 2012); the problem faced by an investor who must liquidate a given basket of assets over a finite time horizon (Schied, Schöneborn & Tehranchi 2010); (Almgren & Lorenz 2007) derive optimal strategies where the execution accelerates when the price moves in the trader's favor, and slows when the price moves adversely;[2]

[2] (Kissell & Malamut 2006) term such adaptive strategies "aggressive-in-the-money"; A "passive-in-the-money" strategy would react oppositely. consider. They assume that the investor's utility has constant absolute risk aversion (CARA) and that the asset prices are given by a very general continuous-time, multi-asset price impact model and show that the investor does no worse if he narrows his search to deterministic strategies. . CARA has exponential utility of the form $u\left(c\right)=1-e^{-\alpha c}$, so that the absolute risk aversion, $A\left(c\right)=-\frac{u''\left(c\right)}{u'\left(c\right)}=\alpha$, a constant. Wikipedia Link on Risk Aversion.

(Schied & Schöneborn 2009) use a stochastic control approach[3], building upon the continuous time model of (Almgren 2003), and show that the value function and optimal control satisfy certain nonlinear parabolic partial differential equations that can be solved numerically. (Kato 2014) develops a mathematical model of optimal execution, by formulating it as a stochastic control problem in the continuous time domain. (Gatheral & Schied 2011) find a closed-form solution for the optimal trade execution strategy in the Almgren-Chriss framework assuming the underlying unaffected stock price (stock price before the impact or before the transaction occurs) process is a GBM; (Schied 2013) investigates the robustness of this strategy with respect to misspecification of the law of the underlying unaffected stock price process. (Guo & Zervos 2015) study the optimal execution problem in the context of a continuous time model with multiplicative price impact, involving singular control rather than absolutely continuous control. [4]

Building on empirical evidence (Lillo, Farmer & Mantegna 2003) that instantaneous market impact is a strongly concave function of the volume, well approximated by a power law function at least for trading rates that are not too high; (Curato, Gatheral & Lillo 2017) find that the discretized cost function exhibits a rugged landscape, with many local minima separated by peaks. (Huberman & Stanzl 2004) provide theoretical arguments showing that in the absence of quasi-arbitrage (availability of a sequence of round-trip trades that generate infinite expected profits with an infinite Sharpe ratio, that is infinite expected profits per unit of risk), permanent price-impact functions must be linear; though empirical investigations suggest that the shape of the limit order book (LOB) can be more complex (Hopman 2007). (Gabaix, Gopikrishnan & Stanley 2006) present a theory in which spikes in trading volume and returns, and hence stock market volatility, are created by a combination of news and the trades of large investors explaining the power law distribution of price impact. (Brunnermeier & Pedersen 2005; Carlin, Lobo & Viswanathan 2007) are extensions to situations with several competing traders, wherein if one trader is forced to liquidate his holdings, other traders also sell creating downward price pressure and buy back the assets later at a lower price.

In contrast to many studies, where the dynamics of the asset price process is taken as a given fundamental, (Obizhaeva & Wang 2013) proposed a market impact model that derives its dynamics from an underlying model of a LOB. In this model, the ask part of the LOB consists of a uniform distribution of shares offered at prices higher than the current best ask price.

[3](Wikipedia Link on Stochastic Control: Stochastic control or stochastic optimal control is a subfield of control theory that deals with the existence of uncertainty either in observations or in the noise that drives the evolution of the system.)

[4] In classical control problems (Shreve 1988), the cumulative displacement of the state, caused by control, is the integral of the control process (or some function of it), and so is absolutely continuous. In impulse control, this cumulative displacement has jumps, between which it is either constant or absolutely continuous. Bounded variation control (defined to include any stochastic control problem in which one restricts the cumulative displacement of the state caused by control to be of bounded variation on finite time intervals) admits both these possibilities and also the possibility that the displacement of the state caused by the optimal control is singularly continuous, at least with positive probability over some interval of time.

(Alfonsi, Fruth & Schied 2010) extend this by allowing for a general shape of the LOB defined via a given density function, which can accommodate empirically observed LOB shapes and obtain a nonlinear price impact of market orders. (Predoiu, Shaikhet & Shreve 2011) derive optimal strategies, (under a general shape of the LOB), that are a mixture of lump purchases and continuous purchases with the rate of purchase set to match the order book resilience. (Fruth, Schöneborn & Urusov 2014) analyze optimal strategies for a risk neutral investor when liquidity varies deterministically (liquidity is time dependent; depth and resilience can be independently time-dependent in contrast to the LOB model of Obizhaeva & Wang 2013) and find that in the case of extreme changes in liquidity, it can even be optimal to completely refrain from trading in periods of low liquidity. Empirical studies based on the LOB model are (Biais, Hillion & Spatt 1995; Potters & Bouchaud 2003; Bouchaud, Gefen, Potters & Wyart 2004; and Weber & Rosenow 2005).

A related strand of literature looks at models of the LOB from the perspective of dealers seeking to submit optimal strategies (maximize the utility of total terminal wealth) of bid and ask orders. (Ho & Stoll 1981) analyze the optimal prices for a monopolistic dealer in a single stock when faced with a stochastic demand to trade, modeled by a continuous time Poisson jump process, and facing return uncertainty, modeled by diffusion processes. (Ho and Stoll 1980), consider the problem of dealers under competition (each dealer's pricing strategy depends not only on his own current and expected inventory position and his other characteristics, but also on the current and expected inventory and other characteristics of the competitor) and show that the bid and ask prices are shown to be related to the reservation (or indifference) prices of the agents.

(Cont, Stoikov & Talreja 2010) describe a stylized model for the dynamics of a limit order book, where the order flow is described by independent Poisson processes and estimate the model parameters from high-frequency order book time-series data from the Tokyo Stock Exchange. (Cont, Kukanov & Stoikov 2014) study the price impact of order book events - limit orders, market orders, and cancellations - using the NYSE Trades and Quotes data for fifty randomly selected stocks. (Avellaneda & Stoikov 2008) combine the utility framework with the microstructure of actual limit order books, as described in the econo-physics literature, to infer reasonable arrival rates of buy and sell orders; (Du, Zhu & Zhao 2016) extend the price dynamics to follow a GBM in which the drift part is updated by Bayesian learning in the beginning of the transaction day to capture the trader's estimate of other traders' target sizes and directions.

(Cont & Kukanov 2017) focus on the order placement problem, which is to choose an order type - market or limit order - and which trading venue(s) to submit it to, when there are multiple alternatives. A numerical algorithm for solving the order placement problem in a general case is provided using a robust modification of the Robbins-Monro stochastic approximation technique (Robbins & Monro 1951; Nemirovski, Juditsky, Lan & Shapiro 2009). (Guo, de Larrard & Ruan 2017) derive optimal placement strategies for both static and dynamic cases (in the static case, as opposed to the dynamic case, a strategy is completely decided before execution takes place, that is at $t = 0$, and is unchanged over the entire order internal), under a correlated random walk model, with mean-reversion for the best ask/bid price.

While our work focuses on separating impact and timing in the (Bertsimas & Lo 1998) framework; a natural and interesting continuation would be to extend this separation to models of the limit order book discussed above (Obizhaeva & Wang 2013).

Models of market impact and the design of better trading strategies are becoming an integral part of the present trend at automation and the increasing use of algorithms. (Jain 2005) assembles the dates of announcement and actual introduction of electronic trading by the leading exchange of 120 countries to examine the long term and medium term impact of automation. He finds that automation of trading on a stock exchange has a long-term impact on listed firms' cost of equity. (Hendershott, Jones & Menkveld 2011) perform an empirical study on New York Stock Exchange stocks and find that algorithmic trading and liquidity are positively related. It is worth noting a contrasting result from an earlier study. (Venkataraman 2001) compares securities on the New York Stock Exchange (NYSE) (a floor-based trading structure with human intermediaries, specialists, and floor brokers) and the Paris Bourse (automated limit-order trading structure). He finds that execution costs might be higher on automated venues even after controlling for differences in adverse selection, relative tick size, and economic attributes. This means fully automated exchanges, which anecdotally seems to be the way ahead, need to take special care to formulate rules, to help liquidity providers better control the risks of order exposure.

What this also means is that, the design of better strategies and models is crucial to survive and thrive in this continuing trend at automation. Our paper aims to fill the gap in existing models of trading costs, which are theoretically elegant but are not readily applicable to real life trading situations, since they do not allow participants to gauge how they are performing in comparison to the other participants with whom they are competing for liquidity. Our models have a strong theoretical foundation but they can be applied to actual trading situations due to the insights they provide to participants. In addition, our numerical framework can be be used to obtain optimal execution schedules under any law of motion of prices.

3 Dynamic Recursive Trading Cost Model

A dynamic programming approach lends itself naturally to modeling optimal execution strategies (Bertsimas & Lo 1998). They start with a simple arithmetic random walk for the law of motion of prices and later extend it to a geometric Brownian motion. Closed form solutions for many scenarios and numerical solutions follow quite easily.

Existing dynamic programming methods to optimizing trading costs and execution scheduling are of limited use to practitioners and traders since they do not provide a way for them to understand how their actions at each stage would affect the price (as opposed to the combined effect of everyone else or the market) and thereby pointing out specific aspects of the system that they can hope to influence. Hence, we start with the benchmark dynamic programming problem and modify the reward function in the Bellman equation to suit our innovation.

3.1 Notation for Optimal Trading using a Dynamic Programming Approach

- $\bar{S}$, the total number of shares that need to be traded.
- T, the total duration of trading.
- N, the number of trading intervals.
- $\tau = T/N$, the length of each trading interval. We assume the time intervals are of the same duration, but this can be relaxed quite easily. In continuous time, this becomes, $N \to \infty, \tau \to 0$.
- The time then becomes divided into discrete intervals, $t_k = k\tau,\ k = 0, ..., N$.
- For simplicity, let time be measured in unit intervals giving, $t = 1, 2, ..., T$.
- S_t, the number of shares acquired in period t at price P_t.
- P_0 can be any reference price or benchmark used to measure the slippage. It is generally taken to be the arrival price or the price at which the portfolio manager would like to complete the purchase of the portfolio.
- Our objective is to formulate a trading trajectory, or a list of total pending shares, $W_1, ..., W_{T+1}$. Here, W_t is the number of units that we still need to trade at time t. This would mean, $W_1 = \bar{S}$ and $W_{T+1} = 0$ implies that $\bar{S}$ must be executed by period T (we note this as an assumption that there will be no unexecuted shares once the total time duration is completed; this would mean that the trading schedule has to be determined to satisfy the constraint that there are no shares left unexecuted at the end of the total trading duration). Clearly, $\bar{S} = \sum_{j=1}^{T} S_j$. This can equivalently be represented by the list of executions completed, $S_1, ..., S_T$. Here, $W_t = W_{t-1} - S_{t-1}$ or $S_{t-1} = W_{t-1} - W_t$ is the number of units traded between times $t-1$ and t at price P_{t-1}. That is we go from W_{t-1} unexecuted shares at time period $t-1$ to W_t remaining shares at time t by filling S_{t-1} shares at price P_{t-1}.W_t and S_t are related as below.

$$W_t = \bar{S} - \sum_{j=1}^{t-1} S_j = \sum_{j=t}^{T} S_j \qquad , t = 1, ..., T.$$

3.2 Benchmark Dynamic Programming Model

This is the simplest scenario where the trader would try to minimize the overall acquisition value of his holdings. This is also the benchmark scenario in (Bertsimas & Lo 1998). In this case, securities are being bought. It is then logical to set a no sales constraint when the objective is to buy securities. The baseline objective function and constraints are written as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} S_t P_t \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S} \, , S_t \geq 0 \, , W_1 = \bar{S}, \; W_{T+1} = 0 \; , \; W_t = W_{t-1} - S_{t-1}$$

The law of motion of price, P_t for the buy scenario can be written as,

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t \; , \theta > 0 \, , E \left[\varepsilon_t \left| S_t, P_{t-1} \right. \right] = 0$$

$$\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

We also follow the convention that the shares are positive when we buy and negative when we sell. The law of motion of price, P_t for the sell scenario then becomes,

$$P_t = P_{t-1} - \theta S_t + \varepsilon_t \; , \theta > 0 \, , E \left[\varepsilon_t \left| S_t, P_{t-1} \right. \right] = 0$$

This price evolution and convention for the buy and sell scenarios ensures that the buyer and the seller have the same price. A trade happens only when the buyer and seller agree upon the price and they both face the same shock in this case. In the rest of the discussion we only consider the price evolution for the buy scenario since this treatment applies with simple modifications when securities are sold.

The law of motion includes two distinct components: the dynamics of P_t in the absence of our trade, (the trades of other may be causing prices to fluctuate) and the impact that our trade of S_t shares has on the execution price P_t. This simple price change relationship assumes that the former component is given by an arithmetic random walk and the latter component is a linear function of trade size so that a purchase of S_t shares may be executed at the prevailing price plus an impact premium of θS_t. Here, θ captures the effect of transaction size on the price. In the absence of this transaction, the price process evolves as a pure arithmetic random walk. This then implies that from any participants view, the sum of all the price movements or the new price levels established by all other participants evolves as a random walk. For simplicity, we ignore the no sales constraint, $S_t \geq 0$.

The Bellman equation is based on the observation that a solution or optimal control $\{S_1^*, S_2^*, ..., S_T^*\}$ must also be optimal for the remaining program at every intermediate time t. That is, for every t, $1 < t < T$ the sequence $\left\{S_t^*, S_{t+1}^*, ..., S_T^*\right\}$ must still be optimal for the remaining program $E_t\left[\sum_{k=t}^{T} P_k S_k\right]$. The below relates the optimal value of the objective function in period t to its optimal value in period $t+1$:

$$V_t\left(P_{t-1}, W_t\right) = \min_{\{S_t\}} E_t\left[P_t S_t + V_{t+1}\left(P_t, W_{t+1}\right)\right]$$

3.3 Terminology and High-level Mathematical Expressions

We now introduce some terminology used throughout the discussion.

1. Total Slippage - The overall price move on the security during the order duration. This is also a proxy for the implementation shortfall (Perold 1988 and Treynor 1981). It is worth mentioning that there are many similar metrics used by various practitioners and this concept gets used in situations for which it is not the best suited (Yegerman and Gillula 2014). While the usefulness of the Implementation Shortfall, or slippage, as a measure to understand the price shortfalls that can arise between constructing a portfolio and while implementing it, is not to be debated, slippage need to be supplemented with more granular metrics when used in situations where the effectiveness of algorithms or the availability of liquidity need to be gauged.

2. Market Impact (MI) - The price moves caused by the executions that comprise the order under consideration. In short, the MI is a proxy for the impact on the price from the liquidity demands of an order. This metric is generally negative (by using a convention to show it as a cost; below we consider it as positive quantity for simplicity) or zero since in most cases, the best impact we can have is usually no impact.

3. Market Timing - The price moves that happen due to the combined effect of all the other market participants during the order duration.

4. Market Impact Estimate (MIE) - This is the estimate of the Market Impact, explained in point two above, based on recent market conditions. The MIE calculation is the result of a simulation which considers the number of executions required to fill an order and the price moves encountered while filling this order, depending on the market micro-structure as captured by the trading volume and the price probability distribution including upticks and down-ticks over the past few days. This simulation can be controlled with certain parameters that dictate the liquidity demanded on the order, the style of trading, order duration, market conditions as reflected by start of trading and end of trading times. In short, the MIE is an estimated proxy for the impact on the price from the liquidity demands of an order. Such an approach holds the philosophical viewpoint that making smaller predictions and considering

their combined effect would result in lesser variance as opposed to making a large prediction; estimations done over a day as compared to estimations over a month, say. A geometrical intuition would be that fitting more lines (or curves) over a set of points would reduce the overall error as compared to fitting lesser number of lines (or curves) over the same set of points. When combining the results of predictions, of course, we have to be mindful of the errors of errors, which can get compounded and lead the results astray, and hence, empirical tests need to be done to verify the suitability of such a technique for the particular situation.

5. Market Timing Estimate (MTE) - This is the estimate of the Market Timing, explained in point three above, based on recent market conditions. The MTE calculation follows from the price volatility and hence longer the duration, we can expect the timing to be higher. It is helpful to consider an upper bound and lower bound for the MTE or in other words a range for the MTE for the duration of trading over which we are estimating the market impact.

6. All these variables are measured in basis points to facilitate ease of comparison and aggregation across different groups. It is possible to measure these in cents per share and also in dollar value or other currency terms.

7. The following equations, expressed in simple mathematical terms to facilitate easier understanding, govern the relationships between the variables mentioned above.

Total Slippage = Market Impact + Market Timing

{Total Price Slippage = Your Price Impact + Price Impact From Everyone Else (Price Drift)}

Market Impact Estimate = Market Impact Prediction = f (Execution Size, Liquidity Demand)

Execution Size = g(Execution Parameters, Market Conditions)

Liquidity Demand = h(Execution Parameters, Market Conditions)

Execution Parameters <->vector comprising (Order Size, Security, Side, Trading Style, Timing Decisions)

Market Conditions <-> vector comprising (Price Movement, Volume Changes, Information Set)

Here, f, g, h are functions. We could impose concavity conditions on these functions, but arguably, similar results are obtained by assuming no such restrictions and fitting linear or non-linear regression coefficients, which could be non-concave or even discontinuous allowing for jumps in prices and volumes. The specific functional forms used could vary across different groups of securities or even across individual securities or even across different time periods for the same security. The crucial aspect of any such estimation is the comparison with the costs on real orders, as outlined earlier. Simpler modes are generally more helpful in interpreting the results and for updating the model parameters. (Hamilton 1994) and (Gujarati 1995) are classic texts on econometric methods and time series analysis that accentuate the need for parsimonious models.

3.4 The Implementation Shortfall

As a refresher, the total slippage or implementation shortfall is derived below with the understanding that we need to use the Expectation operator when we are working with estimates or future prices. (Kissell 2006) provides more details including the formula where the portfolio may be partly executed.

$$\text{Paper Return} = \bar{S}P_T - \bar{S}P_0$$

$$\text{Real Portfolio Return} = \bar{S}P_T - \left(\sum_{t=1}^{T} S_t P_t\right)$$

$$\begin{aligned}
\text{Implementation Shortfall} &= \text{Paper Return} - \text{Real Portfolio Return} \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0
\end{aligned}$$

This can be written as,

$$\begin{aligned}
\text{Implementation Shortfall} &= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0 \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - P_0\left(\sum_{t=1}^{T} S_t\right) \\
&= S_1P_1 + S_2P_2 + ... + S_TP_T - S_1P_0 - S_2P_0 - ... - S_TP_0 \\
&= S_1\left(P_1 - P_0\right) + S_2\left(P_2 - P_0\right) + ... + S_T\left(P_T - P_0\right)
\end{aligned}$$

$$\begin{aligned}
\text{Implementation Shortfall} = \; & S_1\left(P_1 - P_0\right) + \\
& S_2\left(P_2 - P_1\right) + S_2\left(P_1 - P_0\right) + \\
& S_3\left(P_3 - P_2\right) + S_3\left(P_2 - P_1\right) + S_3\left(P_1 - P_0\right) + \\
& ... + \\
& S_T\left(P_T - P_{T-1}\right) + S_T\left(P_{T-1} - P_{T-2}\right) + ... + S_T\left(P_1 - P_0\right)
\end{aligned}$$

The innovation we introduce would incorporate our earlier discussion about breaking the total impact or slippage, Implementation Shortfall, into the part from the participants own decision process, Market Impact, and the part from the decision process of all other participants, Market Timing. This Market Impact, would capture the actions of the participant, since at each stage the penalty a participant incurs should only be the price jump caused by their own trades and that is what any participant can hope to minimize. A subtle point is that the Market Impact portion need only be added up when new price levels are established. If the price moves down and moves back up (after having gone up once earlier and having been already counted in the Impact), we need not consider the later moves in the Market Impact (and hence implicitly left out from the Market Timing as well). This alternate measure would only account for the net move in the prices but would not show the full extent of aggressiveness and the push and pull between market participants and hence is not considered here, though it can be useful to know and can be easily incorporated while running simulations. We discuss two formulations of our measure of the Market Impact in the next two subsections. The reason for calling them simple and complex will become apparent as we continue the discussion.

3.5 Market Impact Simple Formulation

The simple market impact formulation does not consider the impact of the new price level established on all the future trades that are yet to be done. From a theoretical perspective it is useful to study this since it provides a closed form solution and illustrates the immense practical application of separating impact and timing. This approach can be a useful aid in markets that are clearly not trending and where the order size is relatively small compared to the overall volume traded, ensuring that any new price level established does not linger on for too long and prices gets reestablished due to the trades of other participants. This property is akin to checking that shocks to the system do not take long to dissipate and equilibrium levels (or rather new pseudo equilibrium levels) are restored quickly. Our measure of the Market Impact then becomes,

$$\text{Market Impact} = \sum_{t=1}^{T} \left\{ \max\left[\left(P_t - P_{t-1} \right), 0 \right] S_t \right\}$$

The Market Timing is then given by,

$$\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^{T} S_t P_t \right) - \bar{S} P_0 - \sum_{t=1}^{T} \left\{ \max\left[\left(P_t - P_{t-1} \right), 0 \right] S_t \right\}
\end{aligned}$$

For illustration, let us consider some examples,

1. When all the successive price moves are above their corresponding previous price, that is $\max\left[\left(P_t - P_{t-1} \right), 0 \right] = \left(P_t - P_{t-1} \right)$, we have

$$\begin{aligned}
\text{Market Impact} &= \sum_{t=1}^{T} \left\{ \max\left[\left(P_t - P_{t-1} \right), 0 \right] S_t \right\} \\
&= S_1 \left(P_1 - P_0 \right) + S_2 \left(P_2 - P_1 \right) + S_3 \left(P_3 - P_2 \right) + \ \dots \ + S_T \left(P_T - P_{T-1} \right)
\end{aligned}$$

$$
\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0 - S_1\left(P_1 - P_0\right) - S_2\left(P_2 - P_1\right) - S_3\left(P_3 - P_2\right) - \ \dots\ - S_T\left(P_T - P_{T-1}\right) \\
&= S_1 P_0 + S_2 P_1 + S_3 P_2 + \ \dots\ + S_T P_{T-1} - \bar{S}P_0 \\
&= S_2\left(P_1 - P_0\right) + S_3\left(P_2 - P_0\right) + \ \dots\ + S_T\left(P_{T-1} - P_0\right)
\end{aligned}
$$

2. Some of the successive prices are below their corresponding previous price, let us say, $(P_2 < P_1)$ and $(P_3 < P_2)$, we have

$$
\begin{aligned}
\text{Market Impact} &= \sum_{t=1}^{T} \left\{\max\left[\left(P_t - P_{t-1}\right), 0\right] S_t\right\} \\
&= S_1\left(P_1 - P_0\right) + S_2\left(0\right) + S_3\left(0\right) + \ \dots\ + S_T\left(P_T - P_{T-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0 - S_1\left(P_1 - P_0\right) - S_2\left(0\right) - S_3\left(0\right) - \ \dots\ - S_T\left(P_T - P_{T-1}\right) \\
&= S_2 P_2 + S_3 P_3 + S_1 P_0 + S_4 P_3 + S_5 P_4 + \ \dots\ + S_T P_{T-1} - \bar{S}P_0 \\
&= S_2\left(P_2 - P_0\right) + S_3\left(P_3 - P_0\right) + S_4\left(P_3 - P_0\right) + S_5\left(P_4 - P_0\right) + \ \dots\ + S_T\left(P_{T-1} - P_0\right)
\end{aligned}
$$

3.6 Market Impact Complex Formulation

Another measure of the Market Impact can be formulated as below which represents the idea that when a participant seeks liquidity and establishes a new price level, all the pending shares or the unexecuted program is affected by this new price level. This is a more realistic approach since the action now will explicitly affect the shares that are not yet executed. This measure can be written as,

$$
\text{Market Impact} = \sum_{t=1}^{T} \left\{\max\left[\left(P_t - P_{t-1}\right), 0\right] W_t\right\}
$$

The Market Timing is then given by,

$$
\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0 - \sum_{t=1}^{T} \left\{\max\left[\left(P_t - P_{t-1}\right), 0\right] W_t\right\}
\end{aligned}
$$

For illustration, let us consider some examples,

1. When all the successive price moves are above their corresponding previous price, that is $\max\left[\left(P_t - P_{t-1}\right), 0\right] = \left(P_t - P_{t-1}\right)$, we have

$$\begin{aligned}
\text{Market Impact} &= \sum_{t=1}^{T}\left\{\max\left[\left(P_t - P_{t-1}\right), 0\right] W_t\right\} \\
&= W_1\left(P_1 - P_0\right) + W_2\left(P_2 - P_1\right) + W_3\left(P_3 - P_2\right) + \ \ldots\ + W_T\left(P_T - P_{T-1}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0 - W_1\left(P_1 - P_0\right) - W_2\left(P_2 - P_1\right) - W_3\left(P_3 - P_2\right) - \ \ldots\ - W_T\left(P_T - P_{T-1}\right) \\
&= \left[\sum_{t=1}^{T}\left(W_t - W_{t+1}\right) P_t\right] - W_1 P_0 - W_1\left(P_1 - P_0\right) \\
&\quad - W_2\left(P_2 - P_1\right) - W_3\left(P_3 - P_2\right) - \ \ldots\ - W_T\left(P_T - P_{T-1}\right) \\
&= \left(W_1 - W_2\right) P_1 + \left(W_2 - W_3\right) P_2 + \ldots + \left(W_T - W_{T+1}\right) P_T \\
&\quad - W_1 P_0 - W_1\left(P_1 - P_0\right) - W_2\left(P_2 - P_1\right) - W_3\left(P_3 - P_2\right) - \ \ldots\ - W_T\left(P_T - P_{T-1}\right) \\
&= 0
\end{aligned}$$

2. Some of the successive prices are below their corresponding previous price, let us say, $\left(P_2 < P_1\right)$ and $\left(P_3 < P_2\right)$, we have

$$\begin{aligned}
\text{Market Impact} &= \sum_{t=1}^{T}\left\{\max\left[\left(P_t - P_{t-1}\right), 0\right] W_t\right\} \\
&= W_1\left(P_1 - P_0\right) + W_2(0) + W_3(0) + \ \ldots\ + W_T\left(P_T - P_{T-1}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S}P_0 - W_1\left(P_1 - P_0\right) - W_2(0) - W_3(0) - \ \ldots\ - W_T\left(P_T - P_{T-1}\right) \\
&= \left[\sum_{t=1}^{T}\left(W_t - W_{t+1}\right) P_t\right] - W_1 P_0 - W_1\left(P_1 - P_0\right) \\
&\quad - W_2(0) - W_3(0) - \ \ldots\ - W_T\left(P_T - P_{T-1}\right) \\
&= \left(W_1 - W_2\right) P_1 + \left(W_2 - W_3\right) P_2 + \ldots + \left(W_T - W_{T+1}\right) P_T \\
&\quad - W_1 P_0 - W_1\left(P_1 - P_0\right) - W_2(0) - W_3(0) - \ \ldots\ - W_T\left(P_T - P_{T-1}\right) \\
&= -W_2 P_1 + W_2 P_2 - W_3 P_2 + W_3 P_3 \\
&= W_2\left(P_2 - P_1\right) + W_3\left(P_3 - P_2\right)
\end{aligned}$$

3.7 Trading Costs as a Zero Sum Game

A formal study of trading costs in the financial markets using the tools of game theory can lead to many interesting conclusions (Fama 1970 is a discussion of fair games and efficient markets; Kyle 1985, Foster & Viswanathan 1990 solve for the Nash equilibrium when trading is viewed as a game between market makers and traders; Hill 1990 considers transaction costs using a game theoretic model with opportunistic behavior; Klemperer 2004 is an overview of how auctions can explain financial crashes and trading frenzies). Even without a set up specific to game theory (that is the notation, terminology and related paraphernalia, which we will pursue in a later game theory only paper), one of the results we obtain, though fairly evident but perhaps surprising given the extent of trading that takes place in today's markets is that, in any given time period the sum of market impact and the sum of market timing across all market participants equals zero.

This is immediately obvious in the case that there are only two participants (one is the buyer, the other is the seller and without two participants we do not have a market or a trade) and there is only one single interval, since negative implementation shortfall for the buyer shows up as positive implementation shortfall for the seller; the impact for the buyer shows up as timing for the seller and vice versa. We note that the total amount bought in any interval is equal to the total amount sold. When there are more than two participants and multiple intervals, if we consider the actions in each interval and add up the impact and timing figures across everyone, it shows the zero sum nature of the trading game (For different types of zero sum games and methods of solving them, see Brown 1951; Gale, Kuhn & Tucker 1951; Von Neumann & Morgenstern 1953; Von Neumann 1954; Rapoport 1973; Crawford 1974; Laraki & Solan 2005; Hamadène 2006); (Bodie & Taggart 1978; Bell & Cover 1980; Turnbull 1987; Hill 2006; Chirinko & Wilson 2008 consider zero sum games in the financial context). The result holds for both the simple and complex formulations of market impact.

Theorem 1. *Trading costs are a zero sum game. The sum of market impact and market timing across all participants, in any given time interval, should equal zero.*

$$\textit{Total Market Impact} + \textit{Total Market Timing} = 0$$

Proof. See Appendix 11.1.

□

Though we refrain from a longer discussion for the sake of brevity; it should be immediately apparent that the zero sum nature of trading costs is applicable outside the financial markets, to all manner of trades within international / intra-national finance and the exchange of all types of goods and services. Another aspect we point out is the difference in the extent of how much timing and impact might vary between financial markets and trading in other products. The relative ease with which products can be liquidated and / or the extent to which they are either consumption or investment goods, affects this property (Kashyap 2014).

4 Alternative and Practical Dynamic Market Impact Model

We discuss the benchmark law of motion of prices while optimizing the simple and complex market impact formulations in this section. More complex extensions of the law of motion of prices are considered in section 6.

4.1 Simple Formulation of the Benchmark Law of Price Motion

Incorporating the Simple Market Impact formulation from section 3.5, the benchmark objective function and the Bellman equation from section 3.2 can be modified as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} \left\{ \max \left[\left(P_t - P_{t-1} \right), 0 \right] S_t \right\} \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S} \ , S_t \geq 0 \ , W_1 = \bar{S}, \ W_{T+1} = 0 \ , \ W_t = W_{t-1} - S_{t-1}$$

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t \ , \theta > 0 \ , E \left[\varepsilon_t \left| S_t, P_{t-1} \right. \right] = 0 \ , \ \varepsilon_t \sim N \left(0, \sigma_\varepsilon^2 \right)$$

The Bellman equation then becomes,

$$V_t \left(P_{t-1}, W_t \right) = \min_{\{S_t\}} E_t \left[\max \left\{ \left(P_t - P_{t-1} \right), 0 \right\} S_t + V_{t+1} \left(P_t, W_{t+1} \right) \right]$$

One additional constraint that is necessary is to restrict the amount of shares available for trading in any time period, when the price in that time period drops in comparison to the previous time period. The algorithm in section 5 shows how these constrainst can be set. This is a practical consideration, since a drop in price is impact for the sellers and timing for the buyers (as a reminder, we are buyers). Hence when the price decreases in comparison to the previous time period, the amount of shares or liquidity is limited and the seller decides how much to make available. When prices are rising, we can justify not having that criteria, since the buyer can bid up the price and decide how much impact we want to incur. A more thorough approach would ensure that the liquidity follows a process of its own and captures this dynamic of sellers and buyers being able to prop the prices from falling further or rising higher respectively. In the extension we consider in section 6.3, some of these aspects can be factored in.

By starting at the end, (time T) and applying the modified Bellman equation and the law of motion for P_t, the relation between pending and executed shares, and the boundary conditions recursively, the optimal control can be derived as functions of the state variables that characterize the information that the investor must have to make his decision in each period. In particular, the optimal value function, $V_T(\cdots)$, as a function of the two state variables P_{T-1} and W_T is given by,

$$V_T(P_{T-1}, W_T) = \min_{\{S_T\}} E_T\left[\max\left\{(P_T - P_{T-1}), 0\right\} S_T\right]$$

Here, the remaining shares W_{T+1} must be zero since there is no choice but to execute all the remaining shares, W_T. We then have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, W_T) = E_T\left[\max\left\{(\theta W_T + \varepsilon_T), 0\right\} W_T\right]$$

Proposition 1. *The value function for the last but one time period is convex and can be written as,*

$$\begin{aligned} V_{T-1}(P_{T-2}, W_{T-1}) \quad &= \quad \min_{\{S_{T-1}\}} \left[S_{T-1}\sigma_\varepsilon \psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon \psi\left\{\xi(W_{T-1} - S_{T-1})\right\}\right] \\ &\qquad \textit{Here, } \psi(u) = u + \phi(u)/\Phi(u)\ ,\ \xi = \frac{\theta}{\sigma_\varepsilon}, \end{aligned}$$

Also, ϕ and $\mathbf{\Phi}$ are the standard normal Probability Density Function, PDF, and Cumulative Distribution Function CDF, respectively.

Proof. See Appendix 11.2.

□

Figure 3 illustrates the shape of some combinations of the distribution functions that we are working with. For the value function we have, the condition for convexity can be derived as $\theta > (3\sigma_\varepsilon/4)$.

Proposition 2. *The number of shares to be executed in each time period follows a linear law.* $S_{T-1}^* = W_{T-1}^*/2, \quad \ldots \quad, S_{T-K-1}^* = W_{T-K-1}/(K+2)\ , S_{T-K}^* = W_{T-K}/(K+1)$ *and the corresponding value functions are,*

$$V_{T-K-1}(P_{T-K-2}, W_{T-K-1}) = \sigma_\varepsilon W_{T-K-1}\left[\frac{\theta}{\sigma_\varepsilon}\frac{W_{T-K-1}}{(K+2)} + \frac{\phi\left(\frac{\theta}{\sigma_\varepsilon}\frac{W_{T-K-1}}{(K+2)}\right)}{\Phi\left(\frac{\theta}{\sigma_\varepsilon}\frac{W_{T-K-1}}{(K+2)}\right)}\right]$$

Proof. This is shown using induction in Appendix 11.3

□

We can see that a minimum exists at each stage. The simple solution follows from the linear rule where the price impact θS_t does not depend on either the prevailing price, P_{t-1}, or the size of the unexecuted order W_t and hence the price impact function is the same in each period and independent from one period to the next. It is easily shown that, $S_1^* = S_2^* = \cdots = S_T^* = \bar{S}/T$. This simply means that the best execution strategy is simply to divide the total order or the total shares $\bar{S}$ into T equal amounts and trade them at regular intervals. (Bertsimas & Lo 1998) has a more detailed discussion. Supposing a closed form solution was absent, we could approximate the solution (numerically solved) using $S_{T-1}^* \approx \xi_0 + \xi_1 W_{T-1} + \xi_2 \left(W_{T-1}\right)^2$ or $S_{T-1}^* \approx \xi_0 \left(W_{T-1}\right)^{\xi_1}$. We can also set $S_{T-1}^* \approx \omega_1 \left(W_{T-1}\right)$ using any well behaved (continuous and differentiable) function, ω_1. We could also include the last known price, P_{t-1}, or other state variables into the above approximation. We discuss this technique in detail including numerical examples in section 5. This numerical approximation approach is simple to implement and lends itself easily to solutions even in the more complex laws of motion to follow in section 6.

Going forward, to lighten the notion, we will drop the * superscript on the number of shares to be executed in each time period, S_T^*, where there is less likelihood of confusion.

4.2 Complex Formulation of the Benchmark Law of Price Motion

Incorporating the Complex Market Impact formulation from the earlier section 3.6, the objective function and the Bellman equation from section 3.2 can be modified as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} \left\{ \max \left[\left(P_t - P_{t-1}\right), 0 \right] W_t \right\} \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S} \, , S_t \geq 0 \, , W_1 = \bar{S}, \; W_{T+1} = 0 \; , \; W_t = W_{t-1} - S_{t-1}$$

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t \; , \theta > 0 \, , E \left[\varepsilon_t \left| S_t, P_{t-1} \right. \right] = 0 \; , \; \varepsilon_t \sim N \left(0, \sigma_\varepsilon^2 \right)$$

The Bellman equation then becomes,

$$V_t \left(P_{t-1}, W_t\right) = \min_{\{S_t\}} E_t \left[\max \left\{ \left(P_t - P_{t-1}\right), 0 \right\} W_t + V_{t+1} \left(P_t, W_{t+1}\right) \right]$$

The optimal value function, $V_T(\cdots)$, as a function of the two state variables P_{T-1} and W_T is given by,

$$V_T(P_{T-1}, W_T) = \min_{\{S_T\}} E_T\left[\max\left\{(P_T - P_{T-1}), 0\right\} W_T\right]$$

Here, the remaining shares W_{T+1} must be zero since there is no choice but to execute all the remaining shares, W_T. We then have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, W_T) = E_T\left[\max\left\{(\theta W_T + \varepsilon_T), 0\right\} W_T\right]$$

Proposition 3. *The value function for the last but one time period is a convex function with a unique minimum, since it is the sum of the portions shown to be convex above (Proposition 1), another convex function and a linear component.*

$$V_{T-1}(P_{T-2}, W_{T-1}) = \min_{\{S_{T-1}\}}\left[W_{T-1}\sigma_\varepsilon\psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon\psi\left\{\xi(W_{T-1} - S_{T-1})\right\}\right]$$

$$\text{Here, } \psi(u) = u + \phi(u)/\Phi(u)\ ;\ \xi = \frac{\theta}{\sigma_\varepsilon}\ ;\ \text{Note that, } W_{T-1} = S_{T-1} + W_T$$

The number of shares to be executed in subsequent time periods and the corresponding value function are obtained by solving,

$$\begin{aligned} W_{T-1} + \frac{\xi(W_{T-1} - S_{T-1})^2\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} + (W_{T-1} - S_{T-1})\left[\frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})}\right]^2 &= \\ 2(W_{T-1} - S_{T-1}) + \frac{1}{\xi}\frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} + \frac{\xi W_{T-1}S_{T-1}\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + W_{T-1}\left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})}\right]^2 & \end{aligned}$$

Proof. See Appendix 11.4.

□

The simple rule established earlier, $S_{T-1} = W_{T-1}/2$, no longer applies here and we need numerical solutions at each stage. The complexity that gets included in this scenario, when we consider the rest of the unexecuted program into the market impact function, can be seen from this expression. We illustrate numerical techniques for obtaining optimal executions in the next section (5).

5 Numerical Framework for Optimal Execution

Below we develop a numerical framework that can provide optimal executions for any law of motion of prices. We specifically illustrate how we can solve the formulations from section 4 with this numerical technique. It should shortly become clear how this solution technique can be applied under any scenario of price changes including multiple sources of uncertainty. The central idea is similar to the American option pricing methodology (Longstaff & Schwartz 2001) that approximates the ex post realized payoffs from continuation on functions of the values of the state variables. In our case, we use least squares to approximate the conditional expectation of the number of shares to execute as a function of the state variables at each stage. The following points capture a high level essence of the algorithm.

5.1 Optimal Execution Algorithm

1. We create a matrix with the number of columns equal to the number of time periods and number of rows equal to the number of different price paths we desire (total number of simulations we are running). The first column in the matrix corresponds to the starting price, P_0 and the total number of shares to execute, W_1.

2. We then randomly sample the number of shares to execute during the next time period from a uniform distribution. During this process, we can enforce constraints on the minimum or maximum amounts we wish to execute during each time period by including them as the lower and upper limits of the uniform distribution.

3. The number of shares to execute, the price at the start of the time period and the price innovation sampled from another suitable distribution ($\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$ in our case) incorporated into the corresponding law of motion give us the number of shares that still remain to be executed after the end of this time period and the starting price point for the next time period. Any additional sources of uncertainty can be included to obtain the next price level.

4. Continuing this iteratively, we obtain a matrix where each node (row and column) represents a different scenario of price and remaining number of shares to execute before the start of the next time period.

5. Starting from the last time period, at each node, we compute the optimal number of shares to execute during that time period and later ones with complete knowledge of the innovations (ε_t) that unfold on that path, using well-known optimization techniques. For the complex impact function, we use the solnp package in R (Ghalanos, Theussl & Ghalanos 2012; Ye 1988); for the simple impact function, we allocate the remaining shares to the remaining time periods based on whether the corresponding innovations are negative and how negative they are.

(a) Considering the below example of obtaining the optimal executions when we are minimizing the complex impact function under the benchmark law of price motion, we write the objective function as,

$$\min_{\{S_t\}} \left[\sum_{t=1}^{t=T} \left\{ \max\left(\theta S_t + \varepsilon_t, 0\right) \left(\sum_{j=t}^{j=T} S_j \right) \right\} \right]$$

Here, $\sum_{t=1}^{t=T} S_t = W_1$; $S_t, W_1 \geq 0$ and $\theta, \varepsilon_t \in R$, that is they are real numbers. Note that, $P_t - P_{t-1} = \theta S_t + \varepsilon_t$

(b) As an example, for $T = 3$ we get,

$$\min_{\{S_1,S_2,S_3\}} \left[\max\left(\theta S_1 + \varepsilon_1, 0\right)\left(S_1 + S_2 + S_3\right) + \max\left(\theta S_2 + \varepsilon_2, 0\right)\left(S_2 + S_3\right) + \max\left(\theta S_3 + \varepsilon_3, 0\right)\left(S_3\right)\right]$$

(c) For the last time period, the optimal number of shares, $S_3^* = S_3$. When we are time period, $T = 2$,we optimize S_2, S_3 using the Rsolnp library such that the following function is minimized,

$$\min_{\{S_2,S_3\}} \left[\max\left(\theta S_2 + \varepsilon_2, 0\right)\left(S_2 + S_3\right) + \max\left(\theta S_3 + \varepsilon_3, 0\right)\left(S_3\right)\right]$$

Here, $\sum_{t=2}^{t=3} S_t = W_2$; $W_2 = W_1 - S_1$; W_2 would have a different value on each price path simulation or for each row in our matrix.

- Considering the below example of obtaining the optimal executions when we are minimizing the simple impact function under the benchmark law of price motion, we write the objective function as,

$$\min_{\{S_t\}} \left[\sum_{t=1}^{t=T} \left\{ \max\left(\theta S_t + \varepsilon_t, 0\right)\left(S_t\right) \right\} \right]$$

Here, $\sum_{t=1}^{t=T} S_t = W_1$; $S_t, W_1 \geq 0$ and $\theta, \varepsilon_t \in R$, that is they are real numbers. Note that, $P_t - P_{t-1} = \theta S_t + \varepsilon_t$

- As an example, for $T = 3$ we get,

$$\min_{\{S_1,S_2,S_3\}} \left[\max\left(\theta S_1 + \varepsilon_1, 0\right)\left(S_1\right) + \max\left(\theta S_2 + \varepsilon_2, 0\right)\left(S_2\right) + \max\left(\theta S_3 + \varepsilon_3, 0\right)\left(S_3\right)\right]$$

- For the last time period, the optimal number of shares, $S_3^* = S_3$. When we are time period, $T = 2$,we optimize S_2, S_3 such that the following function is minimized,

$$\min_{\{S_2,S_3\}} \left[\max\left(\theta S_2 + \varepsilon_2, 0\right)\left(S_2\right) + \max\left(\theta S_3 + \varepsilon_3, 0\right)\left(S_3\right)\right]$$

Here, $\sum_{t=2}^{t=3} S_t = W_2$; $W_2 = W_1 - S_1$; W_2 would have a different value on each price path simulation or for each row in our matrix. The remaining shares W_2 are distributed to time periods that have negative innovations, starting with earlier time periods, until the execution size times the impact parameter θ plus the innovation equals zero for a particular time period. When this condition is satisfied, we incur zero impact $\{\theta S_1 + \varepsilon_1 = 0 \Rightarrow P_t - P_{t-1} = 0\}$. After the execution size times the impact parameter θ plus the innovation equals zero for all time periods, any further leftover shares are allocated, while giving precedence to earlier time periods and then allocating to subsequent time periods, up-to the maximum execution limit for each time period, since the execution of these shares will cause an equal jump up in the prices (having an equal impact in the objective function) and it is better to execute sooner rather than later.

6. We then run a regression across all the rows in the matrix (this is a cross sectional regression) with the independent variables as the price and the number of shares remaining to be executed before the time period starts and the optimal number of shares to execute during that time period as the dependent variable.

   - We use a regression model such as the one below. It should be clear that we can extend this to purely non-linear regressions or a combination of linear and non-linear components.

$$E\left[S_t \,|W_t, P_{t-1}\right] = \beta_0 + \beta_1 W_t + \beta_2 W_t^2 + \beta_3 P_{t-1} + \beta_4 P_{t-1}^2 + \beta_5 W_t P_{t-1}$$

   For $T = 2$,

$$E\left[S_2 \,|W_2, P_1\right] = \beta_0 + \beta_1 W_2 + \beta_2 W_2^2 + \beta_3 P_1 + \beta_4 P_1^2 + \beta_5 W_2 P_1$$

7. Likewise, we continue backwards in time and obtain regression co-efficients for each time period. The regression co-efficients can then be used to calculate the optimal number of shares before the start of each time period. At each stage, we adjust the number of shares remaining before the time period starts based on the difference between the simulated number of shares to execute and the conditional expected value of the number shares to execute as given by the above regression equation. For $T = 2$, this adjusted number of remaining shares, $\hat{W}_2$, is given by,

$$\hat{W}_2 = S_3 + \left(S_2 - E\left[S_2 \,|W_2, P_1\right]\right)$$

5.2 Sample Results with Mean-Variance of Execution Costs

For the complex impact formulation, the table in (Figure 4) gives the regression coefficients when the number of time periods, $T = 20$ and the total number of shares to execute is $100,000$. Starting with the initial time period in the first row, the optimal executions, the price path and other parameters are also shown. All the parameter values are taken to be the same as the values in (Bertsimas & Lo 1998), to facilitate a proper comparison. (Figure 5) shows the optimal execution schedules under different levels of minimum and maximum number of shares to execute during each time period and different number of price paths or simulation counts.

(Figure 6) compares the average and variance of the total impact costs of our numerical methodology with the benchmark case in (Bertsimas & Lo 1998; also termed the naive strategy), where the solution we get is to execute equal amount of shares in each time period. We report the mean and variance over a simulation sample of 50,000 price paths. We see that the benchmark case has a mean of around 5,262,583 which is comparable to the average execution cost of 5,264,706 using the complex formulation; but the variance is significantly lower using our methodology (769,801,363 in our case versus 1,120,457,643 in the benchmark model). (Figure 6) also reports the multiple of ten percentile values for the executions costs. (Figure 7) shows the histograms of the total costs under the two techniques (the top histogram is for the complex formulation). In addition, our methods are more realistic and adaptive, since the execution amounts change every-time we use it, as the market moves and as our trading progresses. Tailoring it to include additional state variables and capture other sources of uncertainty is relatively straightforward.

Lastly to provide a better understanding of how execution costs change with changes in the different parameters, in (Figure 8) we provide the average of the total execution costs, the simple impact costs, the complex impact costs and the market timing costs across different parameter values, when we are optimizing the complex formulation of the market impact. We impose non-negativity constraints on the execution amounts while calculating the regression co-efficient; later when we use the regression coefficients to calculate execution costs, we remove this restriction for some iterations; this parameter is captured as the maximum and minimum number of shares we can trade in any given time period.

The following values of the parameters are used in the computations: we vary the volatility of the stock price $\sigma = \{0.125, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70, 0.75\}$, the impact parameter, $\theta = \{2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5\}$, the maximum and minimum number of shares we can execute in any given time period, i.e, the liquidity, $\{6666, 13332, 19998, 26664, 3333\}$ and $\{0, -10, 000\}$, and the number of simulations, $\{50000, 20000, 10000, 3000, 2000, 1000\}$. This gives a matrix of summary statistics with around 93 different combinations for which we calculate the regression coefficients, the simple market impact cost, the complex market impact cost and the total execution cost. It is immediately obvious that increasing the impact parameter, θ, leads to an increase in the total executions costs. The increase in the price volatility and the liquidity in each time period do not show such a clear pattern and further investigation is warranted.

But we can expect that the greater uncertainty due to higher price volatility and lack of liquidity, would force participants to trade greater amounts earlier in the trading horizon or as liquidity becomes available.

To calculate the regression coefficients in the quickest possible time, it would be helpful to build a decent amount of computing infrastructure. Since each price path can be developed independently and the only dependence across price paths is while doing the cross-sectional regressions, all the price paths and the optimization at each time period can be done using parallel processing technology. If there are 20 time periods and 1000 price paths, we would need to perform 20,000 Rsolnp optimization calls to compute all the regression coefficients for the complex impact formulation. This is the most time consuming portion of the algorithm and it is highly sensitive to the initial values provided for the routine. The calculation time increases significantly with the number of price paths and time periods; this increase is linear with the number of price paths but it can be more costly to perform the Rsolnp optimizations when the number of time periods increase. To make the calculation engine more robust we also build rudimentary intelligence, such that in case of any interruptions the system will revert back and resume the calculations from the last clean state that was reached. We ran our simulations on an Intel four core windows 10 machine with 4.00 Gigabytes RAM and 2.4 Gigahertz processor speed. In the summary statistics below, we provide the time it takes to calculate all the co-efficients.

To reduce the number of calculations, when we are looking to run optimal executions across hundreds of different securities, we could create groups of securities based on similarties in starting prices, volatilities and other paremeters and compute regression co-efficients for each group separately. Optimizing the simple impact formulation takes considerably less time. The regression co-efficients, optimal executions, executions costs and run times are summarized in (Figures 9, 10, 11). Another alternative to optimizing the complex impact function (instead of the Rsolnp optimization) is to perform a one step ahead optimization. To elaborate on this, at each step, we only look at whether the price went up or down and execute accordingly with full foresight of only one time period. We then run the cross-sectional regressions based on the optimal shares with just one time period look ahead. The regression co-efficients, optimal executions, executions costs and run times are summarized in (Figures 12, 13, 14). It would be prudent to re-calibrate the regression co-efficients periodically across all three formulations.

5.3 Actual Trading Costs Attribution

The following diagram (Figure 15) illustrates the distribution of actual trading costs (These metrics are for live institutional trades from a global sample measured in basis points on the y axis; the x-axis has the costs for two months: April and May 2015; the size of the bubble represents the trade size) based on the our attribution methodology. (Kashyap 2015, 2016) are empirical examples of applying the above methodology to recent market events, wherein, Mincer Zarnowitz type regressions (Mincer & Zarnowitz 1969) are run to establish the accuracy of the estimates. These studies demonstrate the effectiveness of this approach in helping us better understand and analyze real life trading situations.

6 Extensions to the Benchmark Law of Price Motion

6.1 Law of Price Motion with Additional Source of Uncertainty

6.1.1 Simple Formulation

The law of price motion can be changed to include an additional source of uncertainty, X_t, which could represent changing market conditions or private information about the security. We assume that this state variable X_t, is serially-correlated and γ captures its sensitivity to the price movements. Incorporating this, the objective function and the Bellman equation become,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} \{ \max \left[(P_t - P_{t-1}) , 0 \right] S_t \} \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S} \ , S_t \geq 0 \ , W_1 = \bar{S}, \ W_{T+1} = 0 \ , \ W_t = W_{t-1} - S_{t-1}$$

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t \ , \theta > 0 \ , E \left[\varepsilon_t \left| S_t, P_{t-1} \right. \right] = 0$$

$$X_t = \rho X_{t-1} + \eta_t \ , \ \rho \in (-1, 1) \equiv \text{AR(1) Process}$$

$$\varepsilon_t \sim N \left(0, \sigma_\varepsilon^2 \right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N \left(0, \sigma_\eta^2 \right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t \left(P_{t-1}, X_{t-1}, W_t \right) = \min_{\{S_t\}} E_t \left[\max \left\{ (P_t - P_{t-1}) , 0 \right\} S_t + V_{t+1} \left(P_t, X_t, W_{t+1} \right) \right]$$

By starting at the end, (time T) we have,

$$V_T \left(P_{T-1}, X_{T-1}, W_T \right) = \min_{\{S_T\}} E_T \left[\max \left\{ (P_T - P_{T-1}) , 0 \right\} S_T \right]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T \left(P_{T-1}, X_{T-1}, W_T \right) = E_T \left[\max \left\{ (\theta W_T + \varepsilon_T + \gamma X_T) , 0 \right\} W_T \right]$$

Proposition 4. *The number of shares to be executed in each time period follows a linear law.* $S_{T-1} = W_{T-1}/2 \quad \ldots \quad S_{T-K-1} = W_{T-K-1}/\left(K+2\right)$ *and the corresponding value function is*

$$V_{T-K-1}\left(P_{T-K-2}, X_{T-K-2}, W_{T-K-1}\right) = \frac{\theta}{(K+2)} W_{T-K-1}^2 + \alpha_{K+1} W_{T-K-1} + \beta W_{T-K-1} \frac{\phi\left(\frac{\theta W_{T-K-1} + (K+2)\alpha_{K+1}}{(K+2)\beta}\right)}{\Phi\left(\frac{\theta W_{T-K-1} + (K+2)\alpha_{K+1}}{(K+2)\beta}\right)}$$

$$\textit{Here, } \alpha_{K+1} = \gamma \rho X_{T-K-2},\ \beta = \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}$$

Proof. See Appendix 11.5.

□

The simple rule established earlier, $S_{T-1} = W_{T-1}/2$, suffices even here, with a similar reasoning that follows from the independence of the price impact from either the prevailing price or the size of the unexecuted order.

6.1.2 Complex Formulation

Incorporating this additional source of uncertainty into the complex market impact formulation, the objective function and the Bellman equation become,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} \left\{ \max\left[\left(P_t - P_{t-1}\right), 0\right] W_t \right\} \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S}\ , S_t \geq 0\ , W_1 = \bar{S},\ W_{T+1} = 0\ ,\ W_t = W_{t-1} - S_{t-1}$$

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t\ , \theta > 0\ , E\left[\varepsilon_t \left| S_t, P_{t-1}\right.\right] = 0$$

$$X_t = \rho X_{t-1} + \eta_t\ ,\ \rho \in (-1, 1) \equiv \text{AR(1) Process}$$

$$\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N\left(0, \sigma_\eta^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t\left(P_{t-1}, X_{t-1}, W_t\right) = \min_{\{S_t\}} E_t\left[\max\left\{\left(P_t - P_{t-1}\right), 0\right\} W_t + V_{t+1}\left(P_t, X_t, W_{t+1}\right)\right]$$

By starting at the end, (time T) we have,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = \min_{\{S_T\}} E_T\left[\max\left\{\left(P_T - P_{T-1}\right), 0\right\} W_T\right]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T + \varepsilon_T + \gamma X_T\right), 0\right\} W_T\right]$$

Proposition 5. *The number of shares to be executed in each time period and the corresponding value function are obtained by solving,*

$$\begin{aligned}
\theta W_{T-1} + \beta W_{T-1}\left\{\frac{\theta}{\beta}\left[-\frac{\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)\phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)} - \left\{\frac{\phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}\right\}^2\right]\right\} \\
-2\theta\left(W_{T-1} - S_{T-1}\right) - \alpha + \beta\left\{-\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}\right. \\
\left. + \frac{\theta\left(W_{T-1} - S_{T-1}\right)}{\beta}\left[\frac{\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)} + \left\{\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}\right\}^2\right]\right\} \quad = \quad 0
\end{aligned}$$

Proof. See Appendix 11.6.

□

The simple rule established earlier, $S_{T-1} = W_{T-1}/2$, no longer suffices here and we need numerical solutions at each stage of the recursion.

6.2 Linear Percentage Law of Price Motion

6.2.1 Simple Formulation

A law of motion based on an arithmetic random walk has a positive probability of negative prices and it also implies that the Market Impact has a permanent effect on the prices. The other issue is that Market Impact as a percentage of the execution price is a decreasing function of the price level, which is counter-factual. Hence we let the execution price be comprised of two components, a no-impact price $\widetilde{P}_t$, and the price impact Δ_t.

$$P_t = \widetilde{P}_t + \Delta_t$$

The no impact price is the price that would prevail in the absence of any market impact. An observable proxy for this is the mid-point of the bid/offer spread. This is the natural price process and we set it to be a Geometric Brownian Motion.

$$\widetilde{P}_t = \widetilde{P}_{t-1} e^{B_t}$$

$$B_t \sim N\left(\mu_B, \sigma_B^2\right) \equiv \text{IID (Independent Identically Distributed) normal random variable}$$

The price impact Δ_t captures the effect of trade size on the transaction price including the portion of the bid/offer spread. As a percentage of the no-impact price $\widetilde{P}_t$, it is a linear function of the trade size S_t and X_t where as before, X_t is a proxy for private information or market conditions. The parameters θ and γ measure the sensitivity of price impact to trade size and market conditions or private information.

$$\Delta_t = \left(\theta S_t + \gamma X_t\right) \widetilde{P}_t$$

$$X_t = \rho X_{t-1} + \eta_t \ , \ \rho \in (-1, 1)$$

$$\eta_t \sim N\left(0, \sigma_\eta^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

The optimization problem and Bellman equation can be written as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} \left\{ \max\left[\left(P_t - P_{t-1}\right), 0\right] S_t \right\} \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S} \ , S_t \geq 0 \ , W_1 = \bar{S}, \ W_{T+1} = 0 \ , \ W_t = W_{t-1} - S_{t-1}$$

$$V_t\left(P_{t-1}, X_{t-1}, W_t\right) = \min_{\{S_t\}} E_t \left[\max\left\{ \left(P_t - P_{t-1}\right), 0\right\} S_t + V_{t+1}\left(P_t, X_t, W_{t+1}\right) \right]$$

By starting at the end, (time T) we have,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = \min_{\{S_T\}} E_T \left[\max\left\{ \left(P_T - P_{T-1}\right), 0\right\} S_T \right]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T \left[\max\left\{ \left(\widetilde{P}_T \left(1 + \theta W_T + \gamma X_T\right) - P_{T-1} \right), 0 \right\} W_T \right]$$

This involves a normal log-normal mixture and solutions are known for handling this distribution under certain circumstances (Clark 1973; Tauchen & Pitts 1983 ; Yang 2008).

Proposition 6. *The value function is of the form,* $E\left[Y_2\right|Y_2>0]$ *where,* $Y_2\ =\ \left(\widetilde{P}_{T-1}W_Te^{B_T}+\theta W_T^2\widetilde{P}_{T-1}e^{B_T}+\gamma\rho X_{T-1}\widetilde{P}_{T-1}W_Te^{B_T}+\gamma\widetilde{P}_{T-1}W_Te^{B_T}\eta_T-W_TP_{T-1}\right)$. *This can be simplified further to,*

$$E\left[\left(e^XY+k\right)\right|\left(e^XY+k\right)>0\right]$$

$$= \quad k+e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}\left[\left\{\frac{\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}\left\{\mu_Y\left[\Phi\left(-\left[\frac{k+\mu_Y}{\sigma_Y}\right]\right)-\Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)\right]-\frac{\sigma_Y}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{k+\mu_Y}{\sigma_Y}\right)^2}-e^{-\frac{1}{2}\left(\frac{\mu_Y}{\sigma_Y}\right)^2}\right]\right\}\right.$$
$$\left.+\left\{\frac{1-\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{1-\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}\left\{\mu_Y\left[1-\Phi\left(-\left[\frac{k+\mu_Y}{\sigma_Y}\right]\right)\right]+\frac{\sigma_Y}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{k+\mu_Y}{\sigma_Y}\right)^2}\right]\right\}\right]$$

Here, $X\sim N\left(\mu_X,\sigma_X^2\right);Y\sim N\left(\mu_Y,\sigma_Y^2\right);X$ *and* Y *are independent. Also,* $k<0$

Proof. See Appendix 11.7.

□

Clearly, the approach outlined in section 5 to use least squares to approximate the conditional expectation as a function of the state variables at each stage can be easily applied. We can also use other numerical techniques (Miranda & Fackler 2002) or approximations to the error function (Chiani, Dardari & Simon 2003).

6.2.2 Complex Formulation

The optimization problem and Bellman equation for the complex case can be written as,

$$\min_{\{S_t\}}E_1\left[\sum_{t=1}^{T}\left\{\max\left[\left(P_t-P_{t-1}\right),0\right]W_t\right\}\right]$$

$$\sum_{t=1}^{T}S_t=\bar{S}\ ,S_t\geq 0\ ,W_1=\bar{S},\ W_{T+1}=0\ ,\ W_t=W_{t-1}-S_{t-1}$$

$$V_t\left(P_{t-1},X_{t-1},W_t\right)=\min_{\{S_t\}}E_t\left[\max\left\{\left(P_t-P_{t-1}\right),0\right\}W_t+V_{t+1}\left(P_t,X_t,W_{t+1}\right)\right]$$

By starting at the end, (time T) we have,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = \min_{\{S_T\}} E_T\left[\max\left\{\left(P_T - P_{T-1}\right), 0\right\} W_T\right]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T can be arrived similar to the simple formulation in Proposition 6.

6.3 Including Liquidity Constraints

6.3.1 Simple Formulation

A practical limitation that arises when trading is the extent of liquidity that is available at any point in time. This becomes a restriction on the amount of shares tradable in any given interval. Volume can be observed and estimated with a reasonable degree of accuracy. Hence, any measure linking volume to trading costs would be a very practical device. There is a voluminous literature that derives theoretical models and looks at the empirical relationship between volume and prices. (Karpoff 1986; 1987; Gallant, Rossi & Tauchen 1992; Campbell, Grossman & Wang 1993; Wang 1994). We fit a specification similar to the one in (Campbell, Grossman & Wang 1993) wherein the price movements can arise due to changes in future cash flows and investor preferences or the risk aversion. The intuition for this would be that a low return due to a price drop could be caused by an increase in the risk aversion or bad news about future cash flows. Changes in risk aversion cause trading volume to increase while news that is public will already have been impounded in the price and hence will not cause additional trading. Low returns followed by high volume are due to increased risk aversion while low returns and low volume are due to public knowledge of a low level of expectation of future returns. As risk aversion increases, the group of investor still willing to hold the stock require a greater return leading to higher future expected returns. Bad news about future cash flows leads to lower expected returns. This is captured as an inverse relation between auto-correlation of returns and trading volume. The simplification we employ combines the two sources of price changes into one, since what can be observed is only the price return. We note that this can be viewed as an extension of the law of price motion with an additional source of uncertainty. Here, O_t is the total volume traded (market volume) in the interval t. The co-efficient α can be positive or negative, γ is positive and θ continues to be positive.

$$\min_{\{S_t\}} E_1\left[\sum_{t=1}^{T}\left\{\max\left[\left(P_t - P_{t-1}\right), 0\right] S_t\right\}\right]$$

$$\sum_{t=1}^{T} S_t = \bar{S}\ , S_t \geq 0\ , W_1 = \bar{S},\ W_{T+1} = 0\ ,\ W_t = W_{t-1} - S_{t-1}$$

$$P_t = (\alpha + 1) P_{t-1} + \theta S_t P_{t-1} - \gamma\left(O_t - S_t\right) P_{t-1} + \varepsilon_t,\ O_t \geq S_t,\ \beta, \theta > 0\ , \alpha \in (-\infty, \infty)\ , E\left[\varepsilon_t \left| S_t, P_{t-1}\right.\right] = 0$$

$$O_t = \rho O_{t-1} + \eta_t \ , \ \rho \in (-1,1) \equiv \text{AR(1) Process}$$

$$\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N\left(0, \sigma_\eta^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t\left(P_{t-1}, O_{t-1}, W_t\right) = \min_{\{S_t\}} E_t\left[\max\left\{\left(P_t - P_{t-1}\right), 0\right\} S_t + V_{t+1}\left(P_t, O_t, W_{t+1}\right)\right]$$

By starting at the end, (time T) we have,

$$V_T\left(P_{T-1}, O_{T-1}, W_T\right) = \min_{\{S_T\}} E_T\left[\max\left\{\left(P_T - P_{T-1}\right), 0\right\} S_T\right]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

Proposition 7. *The value functions are of the form,* $E\left[Y \middle| Y > 0\right]$ *where,*

$Y = \left(\alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma\rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T\right)$. *For the last and last but one time periods, these can be simplified further to,*

$$V_T\left(P_{T-1}, O_{T-1}, W_T\right) = \left(\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}\,\right) W_T \psi\left(\xi W_T\right) \ , \ \xi W_T = \left(\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma\rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)$$

and

$$V_{T-1}\left(P_{T-2}, O_{T-2}, W_{T-1}\right) = \min_{\{S_{T-1}\}} \quad E_{T-1}\left\{ S_{T-1}\left(\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}\,\right)\right.$$

$$\left[\left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma\rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right) + \frac{\phi\left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma\rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma\rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]$$

$$+ \left(W_{T-1} - S_{T-1}\right)\left(\sqrt{\gamma^2 \left\{P_{T-2}\left(\alpha + 1 + \beta S_{T-1} - \gamma\rho O_{T-2} - \gamma\eta_{T-1}\right) + \varepsilon_{T-1}\right\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}\,\right)$$

$$\left[\left(\frac{\left\{P_{T-2}\left(\alpha + 1 + \beta S_{T-1} - \gamma\rho O_{T-2} - \gamma\eta_{T-1}\right) + \varepsilon_{T-1}\right\}\left\{\alpha + \beta\left(W_{T-1} - S_{T-1}\right) - \gamma\rho^2 O_{T-2} - \gamma\rho\eta_{T-1}\right\}}{\sqrt{\gamma^2\left\{P_{T-2}\left(\alpha + 1 + \beta S_{T-1} - \gamma\rho O_{T-2} - \gamma\eta_{T-1}\right) + \varepsilon_{T-1}\right\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)\right.$$

$$\left.\left. + \frac{\phi\left(\frac{\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}\left\{\alpha+\beta(W_{T-1}-S_{T-1})-\gamma\rho^2 O_{T-2}-\gamma\rho\eta_{T-1}\right\}}{\sqrt{\gamma^2\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}\left\{\alpha+\beta(W_{T-1}-S_{T-1})-\gamma\rho^2 O_{T-2}-\gamma\rho\eta_{T-1}\right\}}{\sqrt{\gamma^2\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}\right]\right\}$$

Here, $\beta = \theta + \gamma$,

Proof. See Appendix 11.8.

□

This requires numerical solutions at each stage of the recursion. A point worth noting is that the simple rule from the earlier linear cases, where the price impact is independent of both the prevailing price and the size of the unexecuted order, no longer applies here. The necessity of having to work with complicated expressions of the sort above, highlights to us the inherent difficulty of making predictions in a complex social system and also that our approach to estimating Market Impact provides a realistic platform upon which further complications, such as working with joint distributions of volume and price, can be built. A key takeaway from this result is that volume can have counter intuitive effects on the trading costs.

6.3.2 Complex Formulation

The optimization problem and Bellman equation for the complex case can be written as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^{T} \left\{ \max\left[\left(P_t - P_{t-1} \right), 0 \right] W_t \right\} \right]$$

$$\sum_{t=1}^{T} S_t = \bar{S} \ , S_t \geq 0 \ , W_1 = \bar{S}, \ W_{T+1} = 0 \ , \ W_t = W_{t-1} - S_{t-1}$$

$$P_t = (\alpha + 1) P_{t-1} + \theta S_t P_{t-1} - \gamma \left(O_t - S_t \right) P_{t-1} + \varepsilon_t, \ O_t \geq S_t, \ \beta, \theta > 0 \ , \alpha \in (-\infty, \infty) \ , E\left[\varepsilon_t \left| S_t, P_{t-1} \right. \right] = 0$$

$$O_t = \rho O_{t-1} + \eta_t \ , \ \rho \in (-1, 1) \equiv \text{AR(1) Process}$$

$$\varepsilon_t \sim N\left(0, \sigma_{\varepsilon}^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N\left(0, \sigma_{\eta}^2\right) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t \left(P_{t-1}, O_{t-1}, W_t \right) = \min_{\{S_t\}} E_t \left[\max\left\{ \left(P_t - P_{t-1} \right), 0 \right\} S_t + V_{t+1} \left(P_t, O_t, W_{t+1} \right) \right]$$

By starting at the end, (time T) we have,

$$V_T \left(P_{T-1}, O_{T-1}, W_T \right) = \min_{\{S_T\}} E_T \left[\max\left\{ \left(P_T - P_{T-1} \right), 0 \right\} W_T \right]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T can be arrived similar to the simple formulation in Proposition 7.

6.4 Trading Costs and Price Spread Sandwich

6.4.1 Simple Formulation

Another useful tool from a trading perspective would be a measure that connects trading costs to the spread, which can be observed. (Roll 1984; Stoll 1989) connect the stock price changes to the bid-offer spread. The spread is determined due to order processing costs, adverse information or inventory holdings costs. The covariance of price changes are related to the covariance of the changes in spread and proportional to the square of the spread, assuming constant spread. A modification with time varying spread can be easily accommodated in the specifications above with an additional source of uncertainty or the linear percentage law of motion. Here, Q_t is the spread at any point in time.

$$P_t = P_{t-1} + \theta S_t + \gamma Q_t + \varepsilon_t \;, \theta > 0 \,, E\left[\varepsilon_t \left| S_t, P_{t-1}\right.\right] = 0$$

$$Q_t = \rho Q_{t-1} + \eta_t \;,\; \rho \in (-1, 1) \equiv \text{AR(1) Process}$$

$\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right) \equiv$ Zero Mean IID (Independent Identically Distributed) random shock or white noise

$\eta_t \sim N\left(0, \sigma_\eta^2\right) \equiv$ Zero Mean IID (Independent Identically Distributed) random shock or white noise

6.4.2 Complex Formulation

This would be analogous to the case in 6.4.1.

7 Conclusions and Possibilities for Future Research

We have developed a trading cost model using dynamic programming that splits the overall price move into the market impact and timing components. The separation of total trading costs into the two components, one of which is directly related to the actions of a participant holds numerous lessons for dealing with complex systems, especially in the social sciences, wherein reducing the complexity by splitting the many sources of uncertainty can lead to better insights in the decision process.

The above decomposition allows us deduce the zero sum game nature of trading costs. In addition, we have develop a powerful numerical technique that can be used under any law of motion of prices and with multiple sources of uncertainty. The starting values we provide for the Rsolnp optimization call can reduce the number of iterations it requires to find the optimal values. Hence, a good extension can be to find better starting values for the optimal value at each time period, based on the innovations and the other parameters.

To ensure that model can be used under different situations, we build upon the benchmark case and introduce more complex formulations of the law of price motion, including a scenario that has multiple sources of uncertainty and consider liquidity or volume constraints. Relating the trading costs to the spread is also easily accomplished. Key improvements to the model and methodology would stem from adding cases where volume and prices are not assumed to be independent. Distributions of prices that are not normal and factor the downward skew in prices might also provide more realistic estimates. Our model takes the prices process as exogenous, interesting continuations can extend the separation of impact and timing to models of the limit order book that endogenously consider the evolution of prices. Again we stress the better insights and understanding that results from using simpler models, but the particulars of the securities being considered might prompt experimenting with some of the more esoteric extensions. We have looked at only discrete time formulations, extensions to continuous time might show interesting theoretical behavior.

A practical way to use these models, would need to factor in the market timing over the duration of trading. Hence, we would need to first get an estimate of the market impact for different time intervals and also calculate the corresponding market timings. Rather than have a single number for the market timing for each impact estimate, it would be more useful to have an upper bound and lower bound or the maximum possible range of the market timing, for each particular time duration. It can be shown that the market timing depends on the price volatility and hence is a key time sensitive variable. Traders can then make a decision regarding which combination of market impact and timing they prefer, since they have more control over the market impact, which is their vessel to navigate the turbulent seas, which is the market timing.

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10 Appendix: Figures

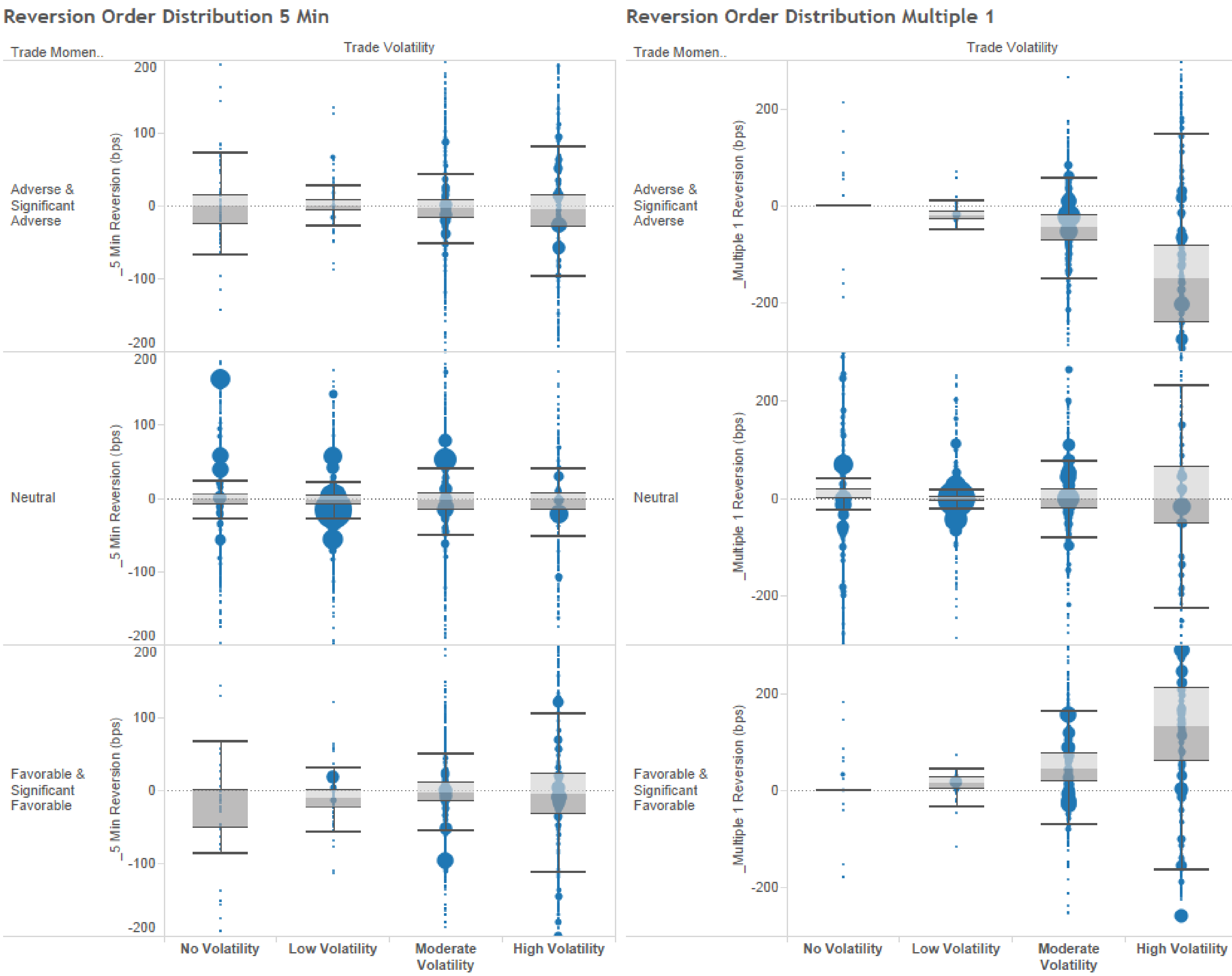


Figure 1: Reversion Distributions by Momentum and Volatility Environments - Shorter Horizon

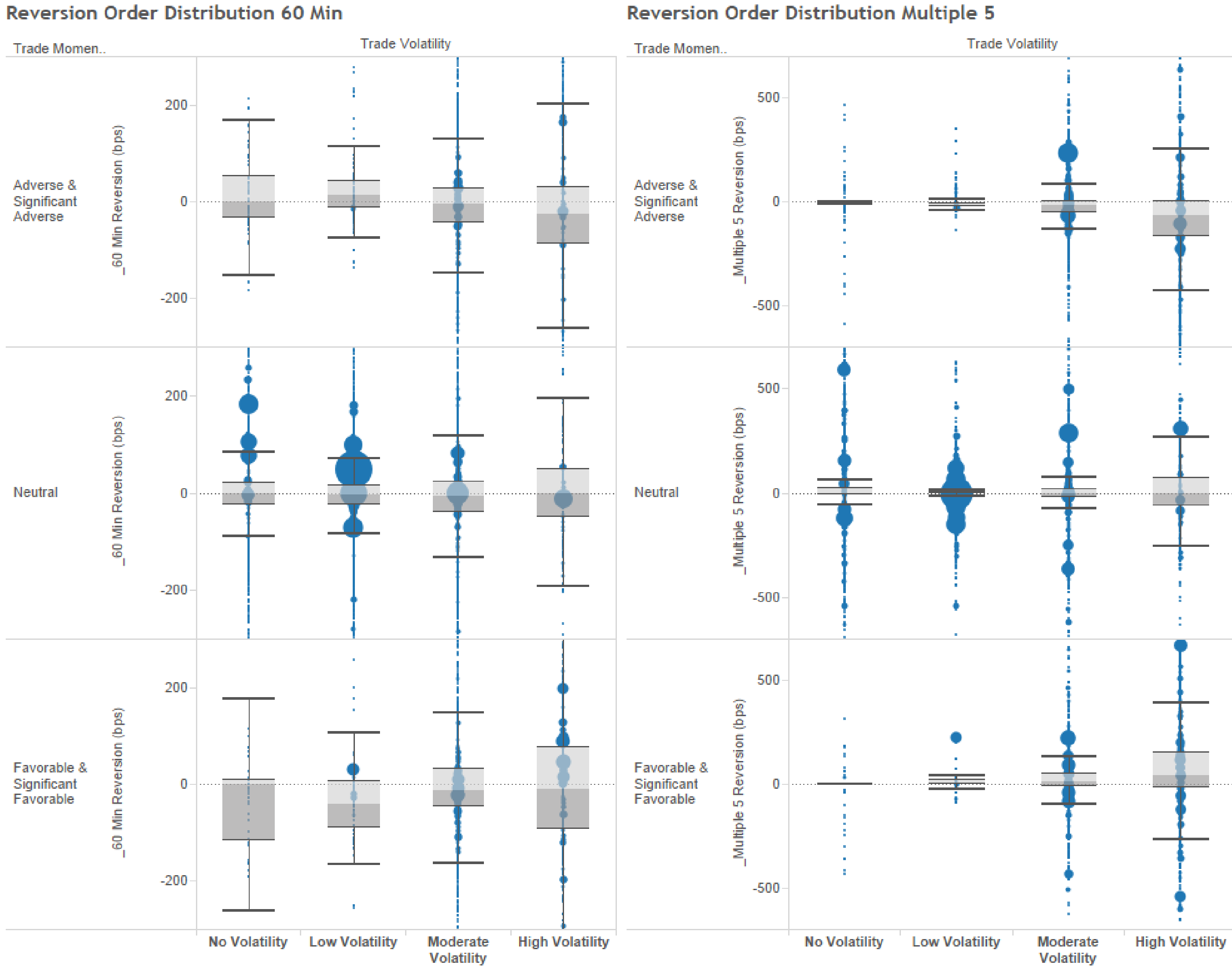


Figure 2: Reversion Distributions by Momentum and Volatility Environments - Longer Horizon

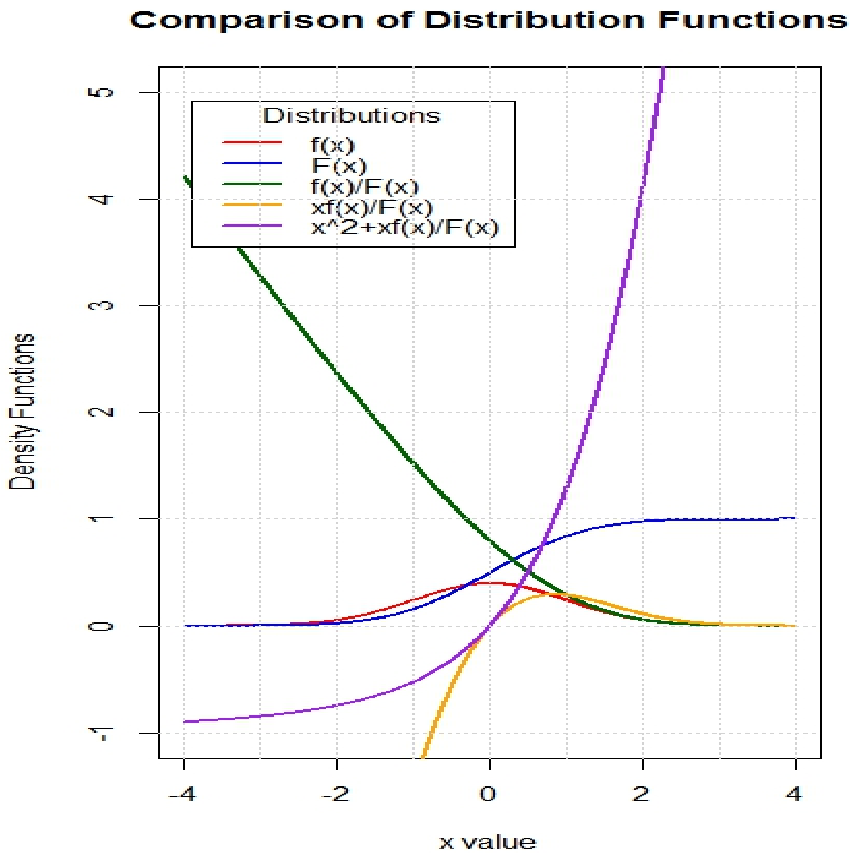


Figure 3: Convexity of Distribution Functions

simulationSampleSize	maxUpShares	maxDownShares	thetaImpact	sigmaStockPrice	marketImpactAvgCost	marketImpactAvgCostCon	totalExecutionAvgCost
50,000	(10,000)	26,664	5.00E-05	0.125	29,890	236,240	5,264,900
intercept	remainingSharesT	priceTMinusOne	remainingSharesTSqr	priceTMinusOneSqr	remainingSharesTtimespri	optimalExecs	prices
(1,371,554.58)	(2.72)	60,690.01	(0.00)	(663.36)	0.06	8,933	50.38
1,350,012.98	(2.59)	(48,456.00)	0.00	434.79	0.05	8,836	50.89
1,442,948.53	(2.56)	(52,255.28)	0.00	473.70	0.05	8,267	51.22
1,319,635.04	(2.05)	(48,273.10)	0.00	442.36	0.04	7,676	51.70
753,257.54	(1.04)	(27,618.52)	0.00	253.91	0.02	7,136	52.03
570,362.25	(0.86)	(20,818.81)	0.00	190.60	0.02	6,607	52.25
234,523.46	(0.31)	(8,520.44)	(0.00)	77.95	0.01	6,026	52.72
265,772.35	(0.37)	(9,668.78)	(0.00)	88.43	0.01	5,427	52.81
133,138.20	(0.10)	(4,838.81)	(0.00)	44.41	0.00	4,835	52.67
54,322.78	0.02	(1,927.99)	(0.00)	17.47	0.00	4,345	52.99
24,216.06	0.06	(837.26)	(0.00)	7.55	0.00	4,025	53.38
138,725.60	(0.20)	(5,020.90)	0.00	45.74	0.01	3,814	53.72
73,760.20	(0.11)	(2,626.08)	0.00	23.67	0.00	3,687	53.97
22,385.36	0.07	(777.32)	0.00	7.02	0.00	3,524	54.29
28,183.57	0.14	(1,006.51)	0.00	9.25	0.00	3,348	54.66
20,781.15	0.17	(749.44)	0.00	7.00	0.00	3,171	54.95
(51,814.38)	0.33	1,905.00	0.00	(17.30)	(0.00)	2,980	55.22
(24,131.15)	0.39	878.80	0.00	(7.86)	(0.00)	2,729	55.32
10,626.82	0.52	(383.74)	0.00	3.53	(0.00)	2,454	55.33
0.00	1.00	(0.00)	0.00	0.00	0.00	2,181	55.41

Figure 4: Complex Regression Co-efficients and Optimal Executions

simulationSampleSize	50,000	20,000	10,000	20,000	10,000	3,000	3,000
maxUpShares	26,664	26,664	26,664	26,664	26,664	6,666	6,666
maxDownShares	-	(10,000)	(10,000)	-	-	-	-
thetaImpact	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.50E-05	5.00E-05
sigmaStockPrice	0.125	0.125	0.125	0.125	0.125	0.5	0.4
marketImpactAvgCost	30,529	29,953	29,817	30,268	30,815	41,311	35,524
marketImpactAvgCostCo	235,703	236,128	236,274	236,212	235,183	308,316	272,492
totalExecutionAvgCost	5,265,304	5,264,832	5,264,844	5,265,150	5,265,254	5,290,421	5,264,101
Optimal Executions 1	8,923	9,065	8,867	8,815	8,933	6,666	6,666
Optimal Executions 2	8,866	8,908	8,866	8,722	8,904	6,666	6,666
Optimal Executions 3	8,348	8,369	8,233	8,243	8,371	6,666	6,666
Optimal Executions 4	7,801	7,753	7,787	7,684	7,911	6,666	6,666
Optimal Executions 5	7,233	7,135	7,200	7,119	7,177	6,666	6,666
Optimal Executions 6	6,653	6,584	6,645	6,665	6,599	6,666	6,666
Optimal Executions 7	6,047	5,999	6,023	6,077	6,072	6,666	6,666
Optimal Executions 8	5,448	5,400	5,394	5,512	5,583	6,666	6,666
Optimal Executions 9	4,956	4,817	4,813	4,938	5,062	6,666	6,666
Optimal Executions 10	4,500	4,264	4,268	4,494	4,702	4,262	4,766
Optimal Executions 11	4,242	3,936	3,939	4,178	4,653	4,839	2,551
Optimal Executions 12	4,117	3,772	3,726	4,217	4,669	5,454	6,666
Optimal Executions 13	4,325	3,680	3,643	4,744	4,683	6,666	6,468
Optimal Executions 14	4,206	3,534	3,502	3,929	4,266	5,674	2,469
Optimal Executions 15	3,700	3,325	3,373	3,639	3,465	5,731	5,641
Optimal Executions 16	2,960	3,155	3,195	2,781	2,913	5,446	4,263
Optimal Executions 17	2,511	2,947	3,026	2,526	2,084	1,934	3,887
Optimal Executions 18	2,354	2,712	2,790	2,175	2,211	-	2,917
Optimal Executions 19	1,321	2,466	2,507	1,212	956	-	378
Optimal Executions 20	1,489	2,178	2,201	2,330	785	-	-

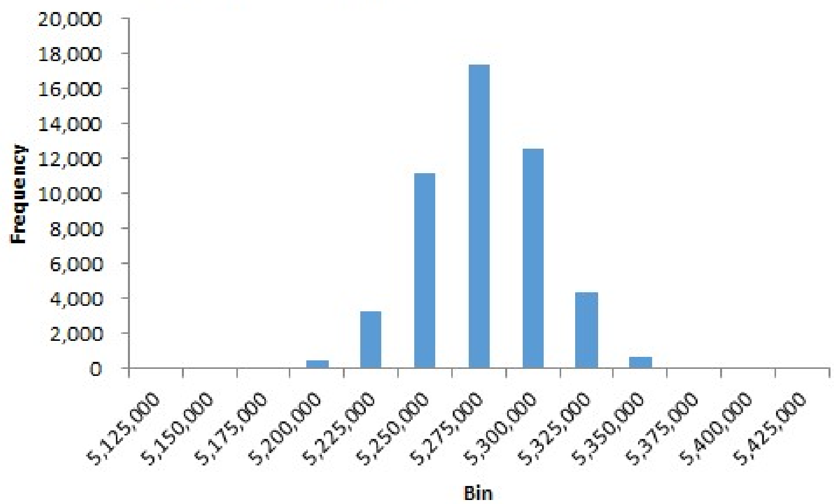
Figure 5: Complex Optimal Executions for Different Parameters

Mean	5,264,706	5,262,583
Variance	769,801,363	1,120,457,643
Std. Deviation	27,745	33,473
Percentile	Complex	Benchmark
0%	5,118,271	5,126,753
10%	5,229,183	5,219,547
20%	5,241,160	5,234,335
30%	5,250,119	5,245,060
40%	5,257,668	5,254,318
50%	5,264,695	5,262,946
60%	5,271,703	5,271,358
70%	5,279,125	5,280,295
80%	5,288,053	5,290,758
90%	5,300,428	5,305,106
100%	5,375,474	5,402,816

Figure 6: Mean / Variance / Percentile Comparison of Total Execution Costs

Complex Formulation

Bin	Frequency
5,125,000	1
5,150,000	0
5,175,000	31
5,200,000	455
5,225,000	3282
5,250,000	11147
5,275,000	17368
5,300,000	12591
5,325,000	4363
5,350,000	715
5,375,000	46
5,400,000	1
5,425,000	0



Benchmark Formulation

Bin	Frequency
5,125,000	0
5,150,000	17
5,175,000	213
5,200,000	1374
5,225,000	4926
5,250,000	11019
5,275,000	14547
5,300,000	11397
5,325,000	4989
5,350,000	1321
5,375,000	174
5,400,000	22
5,425,000	1

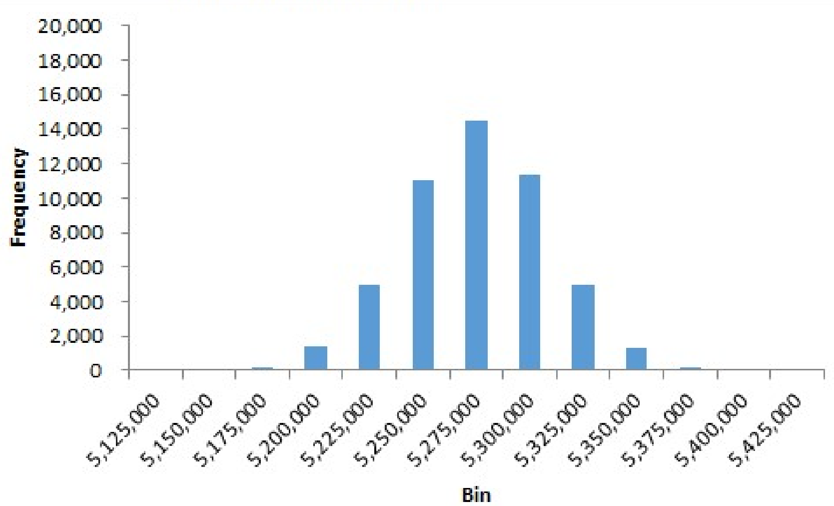


Figure 7: Histogram of Total Execution Costs

simulationSampleSize	maxUpShares	maxDownShares	thetaImpact	sigmaStockPrice	marketImpactAvgCos	marketImpactAvgCos	totalExecutionAvgCos	totalTime	optimalTime	randomTime
10000	26664	-	5.00E-05	0.125	30,815	235,183	5,265,254	54,733	54,151	515
20000	26664	-	5.00E-05	0.125	30,268	236,212	5,265,150	110,128	108,400	1,321
50000	26664	-	5.00E-05	0.125	30,529	235,703	5,265,304	286,536	280,868	4,669
10000	26664	(10,000)	5.00E-05	0.125	29,817	236,274	5,264,844	63,522	62,942	515
20000	26664	(10,000)	5.00E-05	0.125	29,953	236,128	5,264,832	127,294	125,561	1,331
50000	26664	(10,000)	5.00E-05	0.125	29,890	236,240	5,264,900	321,708	316,017	4,829
1000	6666	-	5.00E-05	0.25	33,265	240,695	5,265,550	3,067	3,031	14
2000	6666	-	5.00E-05	0.25	31,573	246,682	5,266,087	6,833	6,744	28
3000	6666	-	5.00E-05	0.25	31,918	243,202	5,264,093	10,161	10,045	78
1000	6666	-	5.00E-05	0.30	33,912	249,515	5,265,142	65,902	65,868	14
2000	6666	-	5.00E-05	0.30	31,818	253,323	5,263,781	7,566	7,498	28
3000	6666	-	5.00E-05	0.30	32,553	252,552	5,265,704	9,012	8,898	76
1000	6666	-	5.00E-05	0.35	33,755	268,001	5,268,507	4,206	4,172	13
2000	6666	-	5.00E-05	0.35	34,260	258,307	5,263,844	6,734	6,663	28
3000	6666	-	5.00E-05	0.35	35,965	255,957	5,265,587	11,049	10,935	71
1000	6666	-	5.00E-05	0.40	36,875	274,768	5,272,076	3,983	3,948	13
2000	6666	-	5.00E-05	0.40	34,276	278,533	5,264,545	7,323	7,255	27
3000	6666	-	5.00E-05	0.40	35,524	272,492	5,264,101	10,819	10,705	72
1000	6666	-	5.00E-05	0.45	36,950	282,624	5,264,777	3,205	3,171	13
2000	6666	-	5.00E-05	0.45	38,322	275,244	5,262,283	7,619	7,552	27
3000	6666	-	5.00E-05	0.45	39,243	276,574	5,267,261	12,128	12,012	71
1000	6666	-	5.00E-05	0.50	40,006	287,137	5,264,134	3,891	3,856	14
2000	6666	-	5.00E-05	0.50	40,253	289,735	5,267,750	8,216	8,147	28
3000	6666	-	5.00E-05	0.50	39,763	290,990	5,269,296	11,281	11,165	73
1000	6666	-	5.00E-05	0.55	41,174	292,383	5,257,389	4,099	4,064	13
2000	6666	-	5.00E-05	0.55	42,078	301,314	5,268,017	7,503	7,436	27
3000	6666	-	5.00E-05	0.55	41,939	299,760	5,268,355	11,975	11,857	72
1000	6666	-	5.00E-05	0.60	44,083	311,440	5,270,659	4,163	4,127	13
2000	6666	-	5.00E-05	0.60	43,398	310,365	5,264,630	7,801	7,730	29
3000	6666	-	5.00E-05	0.60	42,989	317,079	5,269,842	12,030	11,912	71
1000	6666	-	5.00E-05	0.65	45,497	330,801	5,271,688	3,830	3,794	13
2000	6666	-	5.00E-05	0.65	45,212	324,530	5,265,981	7,039	6,970	27
3000	6666	-	5.00E-05	0.65	45,982	322,567	5,268,202	11,004	10,888	72
1000	6666	-	5.00E-05	0.70	44,505	344,219	5,251,540	3,292	3,257	14
2000	6666	-	5.00E-05	0.70	47,155	336,200	5,266,767	8,035	7,965	28
3000	6666	-	5.00E-05	0.70	47,804	335,833	5,268,763	12,156	12,039	73
1000	6666	-	5.00E-05	0.75	49,498	346,341	5,266,310	3,878	3,842	14
2000	6666	-	5.00E-05	0.75	49,563	348,896	5,270,018	7,685	7,615	28
3000	6666	-	5.00E-05	0.75	48,614	349,114	5,262,539	11,542	11,425	71
1000	6666	-	2.50E-05	0.50	29,032	212,243	5,132,818	3,197	3,161	13
2000	6666	-	2.50E-05	0.50	28,958	211,136	5,133,515	6,659	6,588	29
3000	6666	-	2.50E-05	0.50	28,977	208,983	5,132,240	10,002	9,885	73
1000	6666	-	3.00E-05	0.50	31,001	219,045	5,153,839	3,862	3,826	13
2000	6666	-	3.00E-05	0.50	31,323	223,155	5,159,468	7,530	7,459	28
3000	6666	-	3.00E-05	0.50	31,657	222,254	5,160,852	10,363	10,245	73
1000	6666	-	3.50E-05	0.50	32,870	248,228	5,188,292	3,867	3,831	13
2000	6666	-	3.50E-05	0.50	33,093	239,964	5,184,030	6,663	6,594	36
3000	6666	-	3.50E-05	0.50	33,152	238,415	5,187,428	10,619	10,504	71
1000	6666	-	4.00E-05	0.50	35,858	256,927	5,213,321	3,190	3,154	13
2000	6666	-	4.00E-05	0.50	35,402	250,803	5,208,358	6,460	6,388	28
3000	6666	-	4.00E-05	0.50	36,419	254,914	5,216,411	10,834	10,716	77
1000	6666	-	4.50E-05	0.50	37,606	269,173	5,236,560	3,283	3,247	13
2000	6666	-	4.50E-05	0.50	37,735	272,105	5,240,231	6,989	6,920	27
3000	6666	-	4.50E-05	0.50	37,634	271,546	5,240,020	11,289	11,171	72
1000	6666	-	5.50E-05	0.50	40,694	315,390	5,293,801	4,441	4,405	13
2000	6666	-	5.50E-05	0.50	40,324	306,076	5,283,723	7,736	7,658	28
3000	6666	-	5.50E-05	0.50	41,311	308,316	5,290,421	14,869	14,728	82
1000	6666	-	6.00E-05	0.50	43,312	337,195	5,323,706	4,347	4,311	15
2000	6666	-	6.00E-05	0.50	45,304	322,364	5,320,297	8,412	8,341	29
3000	6666	-	6.00E-05	0.50	45,002	324,161	5,319,175	11,950	11,828	73
1000	6666	-	6.50E-05	0.50	48,265	340,347	5,346,085	4,291	4,254	13
2000	6666	-	6.50E-05	0.50	47,115	344,337	5,349,020	8,114	8,044	28
3000	6666	-	6.50E-05	0.50	45,547	353,334	5,349,371	11,595	11,476	71
1000	6666	-	7.00E-05	0.50	51,688	360,537	5,374,780	5,476	5,439	13
2000	6666	-	7.00E-05	0.50	50,857	358,121	5,369,955	7,960	7,889	29
3000	6666	-	7.00E-05	0.50	47,649	365,944	5,368,020	13,226	13,106	71
1000	6666	-	7.50E-05	0.50	53,527	379,429	5,398,777	4,009	3,970	13
2000	6666	-	7.50E-05	0.50	51,551	385,405	5,400,958	8,527	8,456	30
3000	6666	-	7.50E-05	0.50	50,469	390,318	5,401,751	12,352	12,233	101
1000	6666	-	5.00E-05	0.50	39,404	291,529	5,264,851	4,008	3,972	13
2000	6666	-	5.00E-05	0.50	40,369	286,484	5,264,438	8,418	8,347	28
3000	6666	-	5.00E-05	0.50	40,335	286,742	5,267,725	12,040	11,922	76
1000	6666	-	5.00E-05	0.50	38,841	296,903	5,263,748	3,396	3,360	13
2000	6666	-	5.00E-05	0.50	40,625	289,662	5,270,370	8,007	7,936	28
3000	6666	-	5.00E-05	0.50	39,280	291,454	5,266,539	10,355	10,233	71
1000	6666	-	5.00E-05	0.50	40,832	295,350	5,276,438	4,629	4,593	13
2000	6666	-	5.00E-05	0.50	39,847	287,259	5,264,214	7,755	7,675	30
3000	6666	-	5.00E-05	0.50	38,534	300,507	5,269,397	12,139	12,019	82
1000	6666	-	5.00E-05	0.50	39,292	284,062	5,260,138	5,036	5,001	12
2000	6666	-	5.00E-05	0.50	39,444	294,658	5,266,290	7,347	7,277	28
3000	6666	-	5.00E-05	0.50	38,293	290,766	5,262,264	10,693	10,574	74
1000	13332	-	5.00E-05	0.50	59,931	236,672	5,282,005	5,037	4,999	14
2000	13332	-	5.00E-05	0.50	58,243	235,377	5,279,346	8,981	8,908	31
3000	13332	-	5.00E-05	0.50	59,064	232,591	5,277,826	13,592	13,461	73
1000	19998	-	5.00E-05	0.50	64,573	230,049	5,283,347	4,537	4,501	14
2000	19998	-	5.00E-05	0.50	65,416	226,216	5,280,298	9,454	9,382	28
3000	19998	-	5.00E-05	0.50	69,226	224,107	5,283,664	13,567	13,445	71
1000	26664	-	5.00E-05	0.50	63,830	230,422	5,281,811	4,885	4,849	13
2000	26664	-	5.00E-05	0.50	64,061	228,360	5,280,323	7,928	7,856	29
3000	26664	-	5.00E-05	0.50	62,871	228,867	5,278,226	11,658	11,535	72
1000	33330	-	5.00E-05	0.50	63,112	228,762	5,278,294	3,720	3,684	13
2000	33330	-	5.00E-05	0.50	65,582	229,434	5,282,930	7,895	7,822	30
3000	33330	-	5.00E-05	0.50	64,878	226,813	5,280,074	11,791	11,672	72

Figure 8: Complex Executions Costs for Different Parameters

simulationSampleSize	maxUpShares	maxDownShares	thetaImpact	sigmaStockPrice	marketImpactAvgCost	marketImpactAvgCostCo	totalExecutionAvgCost
50,000	(10,000)	26,664	5.00E-05	0.125	59,325	221,153	5,279,818
intercept	remainingSharesT	priceTMinusOne	remainingSharesTSqr	priceTMinusOneSqr	remainingSharesTtimesp	optimalExecs	prices
(4,695,257.69)	3.97	180,917.79	(0.00)	(1,750.41)	(0.06)	13,521	50.59
(780,061.22)	0.33	31,246.54	0.00	(306.81)	(0.01)	13,341	51.34
(1,255,626.42)	1.99	46,580.56	(0.00)	(427.82)	(0.04)	13,520	52.07
461,663.54	(0.22)	(17,370.25)	(0.00)	167.76	0.00	13,329	52.71
520,231.17	(0.53)	(18,919.57)	(0.00)	175.17	0.01	13,014	53.23
(12,337.10)	(0.29)	1,133.44	(0.00)	(14.55)	0.01	11,511	53.96
(1,471,654.07)	3.17	53,371.92	(0.00)	(482.52)	(0.05)	9,703	54.50
1,274,243.58	(1.64)	(46,538.29)	(0.00)	425.81	0.04	6,669	54.98
572,552.21	(0.52)	(20,709.71)	(0.00)	187.92	0.02	4,034	55.43
(166,667.63)	0.90	5,906.11	(0.00)	(51.81)	(0.01)	1,358	55.59
(154,294.81)	0.74	5,581.63	(0.00)	(50.13)	(0.00)	-	55.68
457,107.73	(0.20)	(16,491.66)	(0.00)	148.99	0.01	-	55.54
(375,763.45)	1.08	13,783.01	(0.00)	(126.22)	(0.01)	-	55.40
(99,644.49)	0.48	3,583.91	(0.00)	(32.16)	0.00	-	55.24
35,140.93	0.26	(1,105.89)	(0.00)	8.48	0.01	(62)	55.05
(437,648.49)	1.67	15,895.24	(0.00)	(144.37)	(0.02)	(89)	54.98
(113,921.61)	0.93	4,143.76	(0.00)	(37.80)	(0.01)	(289)	54.92
439,676.22	(0.07)	(16,103.93)	(0.00)	147.10	0.01	(823)	54.76
352,088.51	(0.14)	(12,840.17)	(0.00)	116.31	0.01	(1,593)	54.79
0.00	1.00	(0.00)	0.00	0.00	0.00	2,855	55.03

Figure 9: Simple Regression Co-efficients and Optimal Executions

simulationSampleSize	50,000	20,000	10,000	20,000	10,000	3,000	3000
maxUpShares	26,664	26,664	26,664	26,664	26,664	6,666	6666
maxDownShares	-	(10,000)	(10,000)	-	-	-	-
thetaImpact	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05
sigmaStockPrice	0.125	0.125	0.125	0.125	0.125	0.5	0.5
marketImpactAvgCost	60,095	59,114	59,325	59,904	60,058	38,627	37,398
marketImpactAvgCostCo	220,022	221,422	221,435	220,091	219,832	290,757	309,926
totalExecutionAvgCost	5,279,926	5,279,839	5,280,004	5,279,803	5,279,703	5,263,218	5,266,761
Optimal Executions 1	13,327	13,076	13,302	12,969	13,327	6,666	6,666
Optimal Executions 2	13,360	13,004	13,471	13,423	13,192	6,666	6,666
Optimal Executions 3	13,355	13,407	13,271	13,243	13,370	6,666	6,666
Optimal Executions 4	13,307	13,210	13,338	13,328	13,530	6,666	6,666
Optimal Executions 5	13,646	13,364	12,889	13,882	13,450	6,666	6,666
Optimal Executions 6	12,735	11,844	12,143	12,813	13,047	6,666	6,666
Optimal Executions 7	9,857	9,890	9,517	9,742	9,598	6,666	6,666
Optimal Executions 8	5,688	6,865	6,504	5,858	5,810	6,666	6,666
Optimal Executions 9	2,621	4,235	3,735	2,613	2,537	6,666	6,666
Optimal Executions 10	1,200	1,105	1,830	1,134	1,114	6,666	5,196
Optimal Executions 11	540	-	-	497	530	6,666	2,973
Optimal Executions 12	133	-	-	254	248	6,666	3,361
Optimal Executions 13	128	-	-	118	125	2,669	2,920
Optimal Executions 14	55	-	-	57	52	4,867	3,833
Optimal Executions 15	20	-	(150)	35	40	3,753	3,882
Optimal Executions 16	13	(132)	34	18	16	2,017	2,792
Optimal Executions 17	8	(647)	(56)	9	5	5,638	4,522
Optimal Executions 18	4	(549)	(952)	2	6	1,063	1,601
Optimal Executions 19	2	(1,613)	(1,982)	3	3	-	3,060
Optimal Executions 20	0	2,941	3,106	1	-	-	5,866

Figure 10: Simple Optimal Executions for Different Parameters

simulationSampleSize	maxUpShares	maxDownShares	thetaImpact	sigmaStockPrice	marketImpactAvgCos	marketImpactAvgCos	totalExecutionAvgCos	totalTime	optimalTime	randomTime
10000	26664	-	5.00E-05	0.125	60,058	219,832	5,279,703	4,358	2,840	445
20000	26664	-	5.00E-05	0.125	59,904	220,091	5,279,803	10,136	7,254	1,588
50000	26664	-	5.00E-05	0.125	60,095	220,022	5,279,926	47,844	36,650	6,464
10000	26664	(10,000)	5.00E-05	0.125	59,325	221,435	5,280,004	4,494	3,244	516
20000	26664	(10,000)	5.00E-05	0.125	59,114	221,422	5,279,839	12,494	8,738	1,638
50000	26664	(10,000)	5.00E-05	0.125	59,325	221,153	5,279,818	54,146	42,994	8,935
1000	6666	-	5.00E-05	0.25	34,059	240,259	5,265,436	280	207	17
2000	6666	-	5.00E-05	0.25	33,272	241,003	5,265,184	528	371	31
3000	6666	-	5.00E-05	0.25	34,034	240,904	5,266,031	897	649	70
1000	6666	-	5.00E-05	0.30	35,365	246,139	5,265,461	269	196	17
2000	6666	-	5.00E-05	0.30	34,624	248,585	5,267,481	525	376	31
3000	6666	-	5.00E-05	0.30	35,502	247,976	5,267,220	845	601	68
1000	6666	-	5.00E-05	0.35	36,231	253,347	5,264,414	280	194	16
2000	6666	-	5.00E-05	0.35	36,267	255,066	5,264,650	542	385	39
3000	6666	-	5.00E-05	0.35	36,746	256,858	5,268,587	847	600	68
1000	6666	-	5.00E-05	0.40	36,673	266,753	5,265,970	280	200	17
2000	6666	-	5.00E-05	0.40	38,095	267,232	5,268,099	557	392	33
3000	6666	-	5.00E-05	0.40	37,657	265,402	5,266,952	871	613	70
1000	6666	-	5.00E-05	0.45	39,263	273,869	5,264,299	282	202	18
2000	6666	-	5.00E-05	0.45	38,574	276,120	5,267,345	549	390	34
3000	6666	-	5.00E-05	0.45	39,570	276,621	5,268,262	853	607	69
1000	6666	-	5.00E-05	0.50	40,584	281,994	5,260,744	278	198	16
2000	6666	-	5.00E-05	0.50	40,722	285,091	5,264,270	571	407	34
3000	6666	-	5.00E-05	0.50	40,643	285,314	5,264,862	848	602	69
1000	6666	-	5.00E-05	0.55	42,164	301,144	5,267,698	277	197	16
2000	6666	-	5.00E-05	0.55	42,286	295,073	5,264,038	546	386	36
3000	6666	-	5.00E-05	0.55	42,134	294,904	5,262,421	843	598	69
1000	6666	-	5.00E-05	0.60	43,769	303,101	5,256,581	275	196	16
2000	6666	-	5.00E-05	0.60	43,646	307,588	5,264,025	541	383	33
3000	6666	-	5.00E-05	0.60	44,434	311,200	5,269,456	843	598	69
1000	6666	-	5.00E-05	0.65	45,539	321,392	5,266,402	278	196	16
2000	6666	-	5.00E-05	0.65	45,570	321,925	5,265,666	542	383	34
3000	6666	-	5.00E-05	0.65	45,721	319,464	5,263,125	840	596	69
1000	6666	-	5.00E-05	0.70	46,539	330,396	5,258,610	272	194	16
2000	6666	-	5.00E-05	0.70	47,880	337,482	5,272,622	541	381	33
3000	6666	-	5.00E-05	0.70	47,010	331,465	5,261,213	834	592	68
1000	6666	-	5.00E-05	0.75	48,705	337,184	5,248,309	271	193	16
2000	6666	-	5.00E-05	0.75	49,630	347,551	5,270,653	541	378	33
3000	6666	-	5.00E-05	0.75	48,777	344,417	5,264,322	833	588	68
1000	6666	-	2.50E-05	0.50	28,997	205,690	5,133,981	274	194	16
2000	6666	-	2.50E-05	0.50	29,224	202,741	5,128,284	535	376	33
3000	6666	-	2.50E-05	0.50	29,230	206,048	5,133,908	842	595	69
1000	6666	-	3.00E-05	0.50	30,884	215,720	5,154,363	280	197	16
2000	6666	-	3.00E-05	0.50	31,450	221,411	5,161,504	550	386	33
3000	6666	-	3.00E-05	0.50	31,322	218,614	5,156,513	861	608	70
1000	6666	-	3.50E-05	0.50	33,297	238,074	5,187,674	284	202	17
2000	6666	-	3.50E-05	0.50	33,836	238,541	5,188,258	555	391	33
3000	6666	-	3.50E-05	0.50	34,043	238,984	5,189,570	875	615	70
1000	6666	-	4.00E-05	0.50	35,708	254,414	5,213,472	286	202	17
2000	6666	-	4.00E-05	0.50	36,208	252,802	5,214,121	560	395	34
3000	6666	-	4.00E-05	0.50	36,132	253,892	5,215,187	878	621	71
1000	6666	-	4.50E-05	0.50	37,944	268,849	5,238,801	290	203	17
2000	6666	-	4.50E-05	0.50	38,100	268,344	5,236,588	563	396	34
3000	6666	-	4.50E-05	0.50	37,657	268,101	5,237,067	879	621	72
1000	6666	-	5.50E-05	0.50	43,259	302,506	5,292,924	289	204	17
2000	6666	-	5.50E-05	0.50	43,256	302,320	5,290,154	570	399	34
3000	6666	-	5.50E-05	0.50	43,489	305,825	5,295,528	882	623	71
1000	6666	-	6.00E-05	0.50	44,100	322,999	5,316,876	289	204	17
2000	6666	-	6.00E-05	0.50	45,261	321,679	5,318,535	567	396	34
3000	6666	-	6.00E-05	0.50	45,774	320,071	5,316,255	884	622	71
1000	6666	-	6.50E-05	0.50	47,962	344,774	5,349,486	288	204	17
2000	6666	-	6.50E-05	0.50	48,262	336,725	5,342,211	764	466	34
3000	6666	-	6.50E-05	0.50	48,564	340,099	5,346,866	1,181	858	104
1000	6666	-	7.00E-05	0.50	51,239	362,846	5,376,567	368	258	19
2000	6666	-	7.00E-05	0.50	51,621	360,938	5,374,198	720	494	42
3000	6666	-	7.00E-05	0.50	51,727	362,178	5,376,357	1,124	794	86
1000	6666	-	7.50E-05	0.50	53,889	381,795	5,402,100	387	259	22
2000	6666	-	7.50E-05	0.50	53,487	378,774	5,398,017	578	404	36
3000	6666	-	7.50E-05	0.50	54,232	378,856	5,399,942	857	605	70
1000	6666	-	5.00E-05	0.50	38,403	286,830	5,257,560	275	193	15
2000	6666	-	5.00E-05	0.50	38,242	290,647	5,262,458	545	376	33
3000	6666	-	5.00E-05	0.50	38,627	290,757	5,263,218	930	584	68
1000	6666	-	5.00E-05	0.50	37,776	306,471	5,260,427	380	271	23
2000	6666	-	5.00E-05	0.50	37,695	309,655	5,265,928	592	410	47
3000	6666	-	5.00E-05	0.50	37,398	309,926	5,266,761	857	622	74
1000	6666	-	5.00E-05	0.50	37,750	316,380	5,265,937	272	192	17
2000	6666	-	5.00E-05	0.50	37,746	310,107	5,265,013	522	368	32
3000	6666	-	5.00E-05	0.50	37,150	312,148	5,263,671	810	572	67
1000	6666	-	5.00E-05	0.50	37,358	310,242	5,265,292	266	189	16
2000	6666	-	5.00E-05	0.50	37,145	312,076	5,262,555	518	363	31
3000	6666	-	5.00E-05	0.50	36,996	308,135	5,263,744	808	571	67
1000	13332	-	5.00E-05	0.50	41,007	277,073	5,265,597	253	174	17
2000	13332	-	5.00E-05	0.50	40,779	278,716	5,267,061	525	361	32
3000	13332	-	5.00E-05	0.50	41,290	280,023	5,269,703	819	575	70
1000	19998	-	5.00E-05	0.50	52,315	244,145	5,273,054	261	183	16
2000	19998	-	5.00E-05	0.50	51,757	246,780	5,272,914	516	354	32
3000	19998	-	5.00E-05	0.50	51,078	246,495	5,270,957	784	549	69
1000	26664	-	5.00E-05	0.50	64,172	226,941	5,279,742	278	200	16
2000	26664	-	5.00E-05	0.50	63,644	227,784	5,279,426	502	349	32
3000	26664	-	5.00E-05	0.50	63,720	228,722	5,280,929	779	542	68
1000	33330	-	5.00E-05	0.50	77,596	218,786	5,290,371	261	184	16
2000	33330	-	5.00E-05	0.50	77,288	217,318	5,288,598	504	348	32
3000	33330	-	5.00E-05	0.50	78,132	217,221	5,289,446	789	545	67

Figure 11: Simple Executions Costs for Different Parameters

simulationSampleSize	maxUpShares	maxDownShares	thetaImpact	sigmaStockPrice	marketImpactAvgCost	marketImpactAvgCostComp	totalExecutionAvgCost
50,000	(10,000)	26,664	5.00E-05	0.125	63,365	218,323	5,281,622
intercept	remainingSharesT	priceTMinusOne	remainingSharesTSqr	priceTMinusOneSqr	remainingSharesTtimesprice	optimalExecs	prices
(4,006,778.42)	7.63	145,525.50	0.00	(1,291.75)	(0.16)	14054	50.73
160,427.77	(0.60)	(4,624.58)	(0.00)	33.84	0.01	14573	51.60
(641,255.37)	2.08	22,512.24	(0.00)	(192.51)	(0.04)	14481	52.24
(206,183.10)	0.11	8,183.17	(0.00)	(77.77)	0.00	14444	52.96
(134,639.69)	0.60	5,010.93	(0.00)	(45.58)	(0.00)	13871	53.69
(314,355.73)	1.11	11,610.44	(0.00)	(106.59)	(0.01)	11832	54.21
385,344.34	0.04	(14,000.17)	(0.00)	127.67	0.01	8595	54.55
(263,077.77)	0.92	9,463.99	(0.00)	(84.71)	(0.01)	5074	54.96
(161,853.32)	0.74	5,850.02	(0.00)	(52.59)	(0.00)	2406	54.99
177,454.88	0.55	(6,572.54)	(0.00)	60.99	0.00	670	55.09
(55,060.90)	1.17	1,946.36	(0.00)	(17.08)	(0.01)	0	55.04
42,120.83	(0.67)	(1,581.50)	(0.00)	14.89	0.02	0	55.17
14,910.50	1.10	(567.98)	(0.00)	5.43	(0.01)	0	55.14
(13,442.92)	(0.27)	478.51	(0.00)	(4.24)	0.02	0	55.21
(14,272.08)	2.70	508.57	(0.00)	(4.52)	(0.04)	0	55.30
(8,300.96)	3.18	295.84	(0.00)	(2.63)	(0.05)	0	55.29
(675.90)	0.53	23.39	(0.00)	(0.20)	0.00	0	54.91
2,746.20	4.82	(102.81)	(0.00)	0.96	(0.07)	0	54.89
(2,776.22)	4.80	100.28	0.00	(0.90)	(0.08)	0	54.98
2,637.79	(0.59)	(96.40)	(0.00)	0.88	0.04	0	54.82

Figure 12: One Step Regression Co-efficients and Optimal Executions

simulationSampleSize	50,000	20,000	10,000	20,000	10,000	3,000	3,000
maxUpShares	10,000	(10,000)	(10,000)	10,000	10,000	10,000	10,000
maxDownShares	26,664	26,664	26,664	26,664	26,664	19,998	33,330
thetaImpact	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05	5.00E-05
sigmaStockPrice	0.125	0.125	0.125	0.125	0.125	0.5	0.5
marketImpactAvgCost	63,980	63,390	63,579	63,840	64,389	54,108	78,756
marketImpactAvgCostCor	218,122	218,580	218,465	217,912	217,480	241,746	214,481
totalExecutionAvgCost	5,281,986	5,281,901	5,281,978	5,281,640	5,281,760	5,275,963	5,288,405
Optimal Executions 1	15033	14048	14079	15145	15010	10894	17338
Optimal Executions 2	15098	14474	14338	15297	15246	11313	17608
Optimal Executions 3	14503	14465	14474	14621	14855	11014	17867
Optimal Executions 4	14454	14435	14472	14295	14711	11196	16186
Optimal Executions 5	13682	13912	14045	13644	13666	10731	13218
Optimal Executions 6	11409	11826	11967	11192	11215	10700	9763
Optimal Executions 7	7869	8570	8567	7996	7634	10303	5685
Optimal Executions 8	4451	5061	5052	4439	4305	9012	2311
Optimal Executions 9	2132	2541	2389	2158	2062	6908	24
Optimal Executions 10	978	668	617	994	935	4755	0
Optimal Executions 11	391	0	0	219	361	2430	0
Optimal Executions 12	0	0	0	0	0	744	0
Optimal Executions 13	0	0	0	0	0	0	0
Optimal Executions 14	0	0	0	0	0	0	0
Optimal Executions 15	0	0	0	0	0	0	0
Optimal Executions 16	0	0	0	0	0	0	0
Optimal Executions 17	0	0	0	0	0	0	0
Optimal Executions 18	0	0	0	0	0	0	0
Optimal Executions 19	0	0	0	0	0	0	0
Optimal Executions 20	0	0	0	0	0	0	0

Figure 13: One Step Optimal Executions for Different Parameters

simulationSampleSize	maxUpShares	maxDownShares	thetaImpact	sigmaStockPrice	marketImpactAvgCost	marketImpactAvgCostComplex	totalExecutionAvgCost	totalTime	optimalTime	randomTime
10000	10,000	26664	5.00E-05	0.125	64,389	217,480	5,281,760	4,068	2,649	500
20000	10,000	26664	5.00E-05	0.125	63,840	217,912	5,281,640	11,915	8,987	1,811
50000	10,000	26664	5.00E-05	0.125	63,980	218,122	5,281,986	44,547	35,102	7,112
10000	(10,000)	26664	5.00E-05	0.125	63,579	218,465	5,281,978	4,181	2,953	509
20000	(10,000)	26664	5.00E-05	0.125	63,390	218,580	5,281,901	10,975	7,882	1,592
50000	(10,000)	26664	5.00E-05	0.125	63,365	218,323	5,281,622	47,382	36,199	7,443
1000	10,000	6666	5.00E-05	0.25	26,030	271,006	5,247,096	265	180	16
2000	10,000	6666	5.00E-05	0.25	26,043	269,456	5,243,159	519	348	33
3000	10,000	6666	5.00E-05	0.25	26,188	270,964	5,245,478	805	546	71
1000	10,000	6666	5.00E-05	0.30	27,221	286,428	5,245,256	269	181	16
2000	10,000	6666	5.00E-05	0.30	27,110	289,437	5,244,126	521	348	34
3000	10,000	6666	5.00E-05	0.30	27,273	289,293	5,244,414	807	548	71
1000	10,000	6666	5.00E-05	0.35	29,142	309,438	5,240,102	272	181	17
2000	10,000	6666	5.00E-05	0.35	28,864	311,968	5,248,950	525	349	35
3000	10,000	6666	5.00E-05	0.35	28,554	311,246	5,246,952	809	547	70
1000	10,000	6666	5.00E-05	0.40	29,892	330,283	5,243,872	268	181	17
2000	10,000	6666	5.00E-05	0.40	30,254	330,559	5,245,190	523	348	34
3000	10,000	6666	5.00E-05	0.40	30,436	331,176	5,245,735	814	547	71
1000	10,000	6666	5.00E-05	0.45	32,196	350,997	5,242,119	270	182	17
2000	10,000	6666	5.00E-05	0.45	32,235	353,907	5,248,684	524	348	34
3000	10,000	6666	5.00E-05	0.45	32,074	349,751	5,242,993	802	547	71
1000	10,000	6666	5.00E-05	0.50	33,530	372,820	5,246,145	269	183	17
2000	10,000	6666	5.00E-05	0.50	33,583	372,127	5,241,452	514	349	33
3000	10,000	6666	5.00E-05	0.50	33,770	375,415	5,247,447	799	547	71
1000	10,000	6666	5.00E-05	0.55	36,456	404,560	5,253,281	265	181	17
2000	10,000	6666	5.00E-05	0.55	35,638	399,042	5,245,124	514	349	33
3000	10,000	6666	5.00E-05	0.55	35,886	400,362	5,250,268	806	552	71
1000	10,000	6666	5.00E-05	0.60	37,466	428,419	5,249,848	265	182	17
2000	10,000	6666	5.00E-05	0.60	37,222	420,802	5,246,065	514	349	34
3000	10,000	6666	5.00E-05	0.60	37,295	419,600	5,239,519	800	548	71
1000	10,000	6666	5.00E-05	0.65	38,982	445,473	5,239,783	268	182	17
2000	10,000	6666	5.00E-05	0.65	39,879	443,432	5,244,730	515	349	34
3000	10,000	6666	5.00E-05	0.65	38,867	441,700	5,239,040	804	550	71
1000	10,000	6666	5.00E-05	0.70	40,708	474,047	5,253,985	266	182	17
2000	10,000	6666	5.00E-05	0.70	41,011	466,455	5,240,444	518	350	33
3000	10,000	6666	5.00E-05	0.70	40,551	464,448	5,236,778	802	548	71
1000	10,000	6666	5.00E-05	0.75	43,036	493,854	5,248,876	266	182	17
2000	10,000	6666	5.00E-05	0.75	43,087	493,433	5,244,963	518	351	34
3000	10,000	6666	5.00E-05	0.75	43,027	497,000	5,254,102	809	550	71
1000	10,000	6666	2.50E-05	0.50	26,045	304,403	5,122,817	270	186	17
2000	10,000	6666	2.50E-05	0.50	26,482	306,716	5,126,555	517	350	34
3000	10,000	6666	2.50E-05	0.50	26,083	303,017	5,119,732	805	549	71
1000	10,000	6666	3.00E-05	0.50	27,772	316,423	5,146,908	269	182	17
2000	10,000	6666	3.00E-05	0.50	27,901	321,736	5,151,447	518	350	34
3000	10,000	6666	3.00E-05	0.50	27,859	319,527	5,149,287	806	550	71
1000	10,000	6666	3.50E-05	0.50	28,110	324,901	5,161,258	267	182	17
2000	10,000	6666	3.50E-05	0.50	28,694	330,643	5,170,459	521	351	34
3000	10,000	6666	3.50E-05	0.50	28,903	331,259	5,171,786	809	553	72
1000	10,000	6666	4.00E-05	0.50	30,815	345,275	5,193,502	269	183	17
2000	10,000	6666	4.00E-05	0.50	30,346	344,807	5,196,725	520	351	34
3000	10,000	6666	4.00E-05	0.50	30,360	343,031	5,192,854	807	551	71
1000	10,000	6666	4.50E-05	0.50	33,315	366,101	5,228,437	272	183	17
2000	10,000	6666	4.50E-05	0.50	31,984	359,369	5,222,287	519	351	34
3000	10,000	6666	4.50E-05	0.50	31,996	361,363	5,220,067	810	552	71
1000	10,000	6666	5.50E-05	0.50	34,909	386,365	5,261,431	268	182	17
2000	10,000	6666	5.50E-05	0.50	35,446	395,217	5,273,920	522	351	34
3000	10,000	6666	5.50E-05	0.50	35,357	387,390	5,267,538	807	551	71
1000	10,000	6666	6.00E-05	0.50	36,171	397,189	5,283,453	270	184	17
2000	10,000	6666	6.00E-05	0.50	36,877	407,699	5,293,455	521	351	34
3000	10,000	6666	6.00E-05	0.50	37,034	406,101	5,294,681	809	549	70
1000	10,000	6666	6.50E-05	0.50	39,541	423,697	5,324,799	269	183	17
2000	10,000	6666	6.50E-05	0.50	38,934	422,837	5,320,228	522	351	34
3000	10,000	6666	6.50E-05	0.50	38,473	422,973	5,319,566	811	553	72
1000	10,000	6666	7.00E-05	0.50	40,454	439,402	5,342,942	273	183	17
2000	10,000	6666	7.00E-05	0.50	40,805	437,677	5,344,658	536	360	35
3000	10,000	6666	7.00E-05	0.50	40,106	437,473	5,342,549	826	564	74
1000	10,000	6666	7.50E-05	0.50	42,473	454,576	5,366,627	274	185	17
2000	10,000	6666	7.50E-05	0.50	42,175	453,460	5,366,082	487	328	36
3000	10,000	6666	7.50E-05	0.50	42,307	457,798	5,369,656	869	540	72
1000	20,000	6666	5.00E-05	0.50	34,466	372,764	5,249,949	305	211	23
2000	20,000	6666	5.00E-05	0.50	34,497	372,124	5,250,406	503	338	33
3000	20,000	6666	5.00E-05	0.50	33,826	369,301	5,242,620	778	526	68
1000	30,000	6666	5.00E-05	0.50	33,549	368,416	5,252,931	260	175	16
2000	30,000	6666	5.00E-05	0.50	34,667	370,448	5,251,108	506	337	33
3000	30,000	6666	5.00E-05	0.50	34,828	372,084	5,256,929	788	527	68
1000	40,000	6666	5.00E-05	0.50	34,491	370,124	5,250,662	262	177	17
2000	40,000	6666	5.00E-05	0.50	34,331	371,333	5,252,584	504	337	33
3000	40,000	6666	5.00E-05	0.50	34,773	371,911	5,254,780	786	537	69
1000	50,000	6666	5.00E-05	0.50	34,416	371,761	5,249,350	263	178	17
2000	50,000	6666	5.00E-05	0.50	34,795	372,850	5,259,548	506	340	33
3000	50,000	6666	5.00E-05	0.50	34,735	371,822	5,254,589	803	544	69
1000	10,000	13332	5.00E-05	0.50	41,188	282,743	5,271,494	263	177	17
2000	10,000	13332	5.00E-05	0.50	41,582	280,088	5,269,434	511	342	33
3000	10,000	13332	5.00E-05	0.50	41,608	278,267	5,268,825	10,858	528	68
1000	10,000	19998	5.00E-05	0.50	54,291	242,346	5,277,672	227	156	16
2000	10,000	19998	5.00E-05	0.50	53,931	243,347	5,277,206	516	343	29
3000	10,000	19998	5.00E-05	0.50	54,108	241,746	5,275,963	954	630	71
1000	10,000	26664	5.00E-05	0.50	64,914	224,201	5,279,376	317	217	21
2000	10,000	26664	5.00E-05	0.50	66,519	225,940	5,283,042	644	423	37
3000	10,000	26664	5.00E-05	0.50	65,948	223,143	5,280,231	924	636	87
1000	10,000	33330	5.00E-05	0.50	79,931	218,713	5,294,371	327	209	19
2000	10,000	33330	5.00E-05	0.50	75,989	213,806	5,283,930	660	449	44
3000	10,000	33330	5.00E-05	0.50	78,756	214,481	5,288,405	1,025	700	95

Figure 14: One Step Executions Costs for Different Parameters

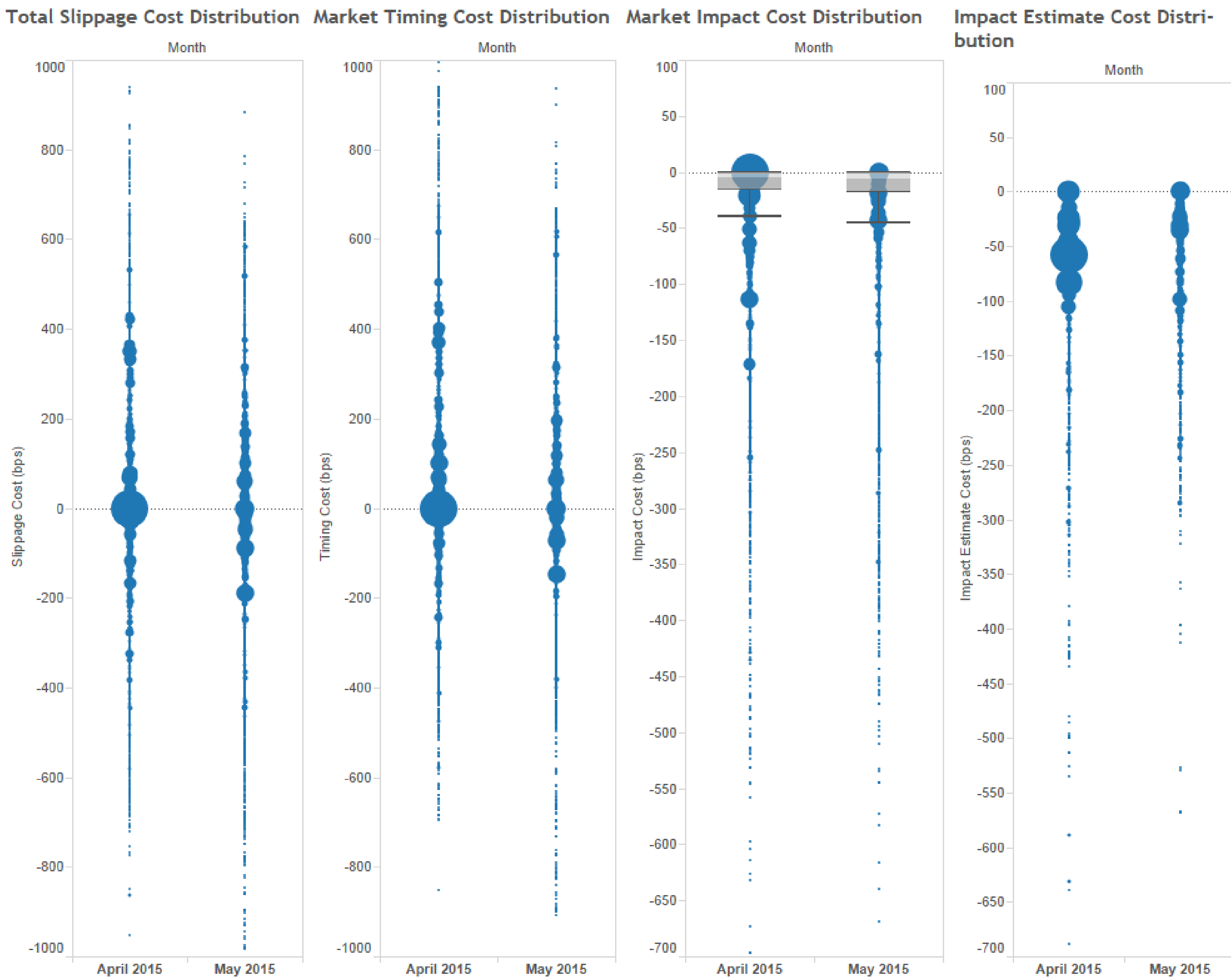


Figure 15: Actual Trading Cost Distributions

11 Appendix of Proofs

11.1 Proof of Theorem 1

Lemma 1. *We first consider the simple formulation, with one interval and two market participants,*

Proof. For the buyer we have,

$$\text{Market Impact} = \{\max\left[\left(P_t - P_{t-1}\right), 0\right] S_t\}$$

$$\begin{aligned}\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\ &= (S_t P_t) - S_t P_0 - \{\max\left[(P_t - P_{t-1}), 0\right] S_t\}\end{aligned}$$

Here, we use the definition of Implementation Shortfall after adapting it to the one interval case,

$$\begin{aligned}\text{Implementation Shortfall} &= \text{Paper Return} - \text{Real Portfolio Return} \\ &= \left(\sum_{t=1}^{T} S_t P_t\right) - \bar{S} P_0 = (S_t P_t) - S_t P_0\end{aligned}$$

Similarly we have for the seller (noting that the drop in prices is detrimental to the intended outcome and changing the sign accordingly),

$$\text{Market Impact} = \{\max\left[(P_{t-1} - P_t), 0\right] S_t\}$$

$$\begin{aligned}\text{Market Timing} &= -\text{Implementation Shortfall} - \text{Market Impact} \\ & S_t P_0 - (S_t P_t) - \{\max\left[(P_{t-1} - P_t), 0\right] S_t\}\end{aligned}$$

If $(P_t > P_{t-1})$,

For the buyer,

$$\text{Market Impact} = (P_t - P_{t-1}) S_t$$

$$\text{Market Timing} = (S_t P_t) - S_t P_0 - (P_t - P_{t-1}) S_t = (P_{t-1} - P_0) S_t$$

For the seller,

$$\text{Market Impact} = 0$$

$$\text{Market Timing} = S_t P_0 - (S_t P_t) - 0 = (P_0 - P_t) S_t$$

Sum of the impact and timing across both the participants,

$$\text{Total Market Impact} = (P_t - P_{t-1}) S_t$$

$$\text{Total Market Timing} = (P_{t-1} - P_0) S_t + (P_0 - P_t) S_t = (P_{t-1} - P_t) S_t$$

$$\text{Total Market Impact} + \text{Total Market Timing} = 0$$

If $(P_t < P_{t-1})$,

For the buyer,

$$\text{Market Impact} = 0$$

$$\text{Market Timing} = (S_t P_t) - S_t P_0 - 0 = (P_t - P_0) S_t$$

For the seller,

$$\text{Market Impact} = (P_{t-1} - P_t) S_t$$

$$\begin{aligned}\text{Market Timing} &= -\text{Implementation Shortfall} - \text{Market Impact} \\ & S_t P_0 - (S_t P_t) - (P_{t-1} - P_t) S_t = S_t P_0 - S_t P_{t-1}\end{aligned}$$

Sum of the impact and timing across both the participants,

$$\text{Total Market Impact} = (P_{t-1} - P_t) S_t$$

$$\text{Total Market Timing} = (P_0 - P_{t-1}) S_t + (P_t - P_0) S_t = (P_t - P_{t-1}) S_t$$

$$\text{Total Market Impact} + \text{Total Market Timing} = 0$$

It should be clear that this holds for all non-zero positive values of prices and number of shares which can include zero, that is $\forall P_{t\in\{t=0,1,2,\ldots,T\}} \in (0,\infty)$ and $\forall S_t \in [0,\infty)$

□

Lemma 2. *We next consider the simple formulation, with multiple intervals and multiple participants.*

1. We argue that this scenario with multiple intervals and multiple participants can be reduced to an amalgamation of the above case (Lemma 1) with a single interval and two participants.

2. Our definition of an interval is such that during each interval, only one exchange happens between buyer and seller for a total of two participants, with the sum of market impact and market timing being equal to zero.

3. To convince us that such an interval exists, we reason as follows: when multiple exchanges happen

during an interval, we split the interval into sub-intervals such that only one exchange happens in each interval. If multiple exchanges happen simultaneously, they can be viewed as one exchange by combining all the buy trades on one side against the sell trades on the other side.

4. The sum of many such individual intervals also have the same property (by mathematical induction), wherein the sum of market impact and market timing equals zero, which follows from (Lemma 1). It should be clear that this holds for all non-zero positive values of prices and number of shares which can include zero, that is $\forall P_{t \in \{t=0,1,2,\ldots,T\}} \in (0, \infty)$ and $\forall S_t \in [0, \infty)$ across all the intervals considered.

Lemma 3. *Lastly, we consider the complex formulation with multiple intervals and multiple participants.*

1. We argue that the complex formulation scenario with multiple intervals and multiple participants can be reduced to an amalgamation of the above two cases (Lemma 1, 2).

2. Any shares unexecuted by the end of a certain time interval will need to be executed before the end of the total time duration available for trading, since we note that by assumption, there will be no unexecuted shares once the total time duration is completed.

3. We apply Lemma 1 to the sum of impact and timing for the shares executed at the last time interval, making this sum zero. We then consider the last interval and the interval before that together and apply Lemma 2 to these two intervals, which gives the sum of impact and timing across both these intervals as zero.

4. We can then include additional intervals towards the beginning of the trading duration and deduce that the sum of impact and timing across the new interval and the already aggregated intervals is zero using mathematical induction. It should be clear that this holds for all non-zero positive values of prices and number of shares which can include zero, that is $\forall P_{t \in \{t=0,1,2,\ldots,T\}} \in (0, \infty)$ and $\forall S_t \in [0, \infty)$ across all the intervals considered.

5. Hence, by considering the shares executed in the last interval and successively including the intervals before that, we get that the corresponding sum of market impact and market timing equals zero. The result is that we have at the end of the total trading duration after aggregating across all the individual intervals, the sum of total market impact and total market timing being equal to zero. This completest the proof of Theorem 1.

11.2 Proof of Proposition 1

Proof. We have from the value function for the last time period.

$$V_T\left(P_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T + \varepsilon_T\right), 0\right\} W_T\right]$$

$$V_T\left(P_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T^2 + W_T\varepsilon_T\right), 0\right\}\right]$$

$$V_T\left(P_{T-1}, W_T\right) = E_T\left[\left.\left(\theta W_T^2 + W_T\varepsilon_T\right)\right| \left(\theta W_T^2 + W_T\varepsilon_T\right) > 0\right]$$

$$\{\because\ E\left[max\left(X, c\right)\right] = E\left[X\left|X > c\right.\right] Pr\left[X > c\right] + E\left[c\left|X \leq c\right.\right] Pr\left[X \leq c\right]\ \}$$

This is of the form, $E\left[Y\right| Y > 0]$ where, $Y = \left(\theta W_T^2 + W_T\varepsilon_T\right)$. We then need to calculate,

$$E_T\left[\left.\left(\theta W_T^2 + W_T\sigma_\varepsilon Z\right)\right| Z > \left(-\frac{\theta W_T}{\sigma_\varepsilon}\right)\right] \quad , where\ Z \sim N\left(0, 1\right)$$

$$\left[\because\ Y \sim N\left(\theta W_T^2, W_T^2\sigma_\varepsilon^2\right) \equiv Y \sim N\left(\mu, \sigma^2\right) \Rightarrow Y = \mu + \sigma Z\ ;\ Y > 0 \Rightarrow Z > -\mu/\sigma\right]$$

We have for every standard normal distribution, Z, and for every u, $Pr\left[Z > -u\right] = Pr\left[Z < u\right] = \mathbf{\Phi}\left(u\right)$. Here, ϕ and $\mathbf{\Phi}$ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E\left[Z\right| Z > -u] &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[\int_{-u}^{\infty} t\phi\left(t\right) dt\right] \\ &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[-\phi\left(t\right)\Big|_{-u}^{\infty}\right] = \frac{\phi\left(u\right)}{\mathbf{\Phi}\left(u\right)} \end{aligned}$$

$$\left[\because \int t\phi\left(t\right) dt = \int t\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2} dt = \int\left\{\frac{d-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}}{dt}\right\} dt\right]$$

Hence we have,

$$\begin{aligned} E\left[Y\right| Y > 0] &= \mu + \sigma E\left[Z\right| Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma\phi\left(\mu/\sigma\right)}{\mathbf{\Phi}\left(\mu/\sigma\right)} \end{aligned}$$

Setting, $\psi\left(u\right) = u + \phi\left(u\right)/\Phi\left(u\right)$,

$$E\left[Y\right| Y > 0] = \sigma\psi\left(\mu/\sigma\right)$$

$$V_T\left(P_{T-1},W_T\right) = E_T\left[\left(\theta W_T^2+W_T\varepsilon_T\right)\middle|\left(\theta W_T^2+W_T\varepsilon_T\right)>0\right]$$

$$= W_T\sigma_\varepsilon\left[\frac{\theta W_T}{\sigma_\varepsilon}+\frac{\phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)}\right]=\sigma_\varepsilon W_T\psi\left(\xi W_T\right)\ ,\ \xi=\frac{\theta}{\sigma_\varepsilon}$$

In the next to last period, $T-1$, the Bellman equation is,

$$V_{T-1}\left(P_{T-2},W_{T-1}\right)=\min_{\{S_{T-1}\}}E_{T-1}\left[\max\left\{\left(P_{T-1}-P_{T-2}\right),0\right\}S_{T-1}+V_T\left(P_{T-1},W_T\right)\right]$$

$$=\min_{\{S_{T-1}\}}E_{T-1}\left[\max\left\{\left(\theta S_{T-1}+\varepsilon_{T-1}\right),0\right\}S_{T-1}+V_T\left(P_{T-2}+\theta S_{T-1}+\varepsilon_{T-1},W_{T-1}-S_{T-1}\right)\right]$$

$$=\min_{\{S_{T-1}\}}\left\{S_{T-1}\sigma_\varepsilon\left[\frac{\theta S_{T-1}}{\sigma_\varepsilon}+\frac{\phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}\right]+\left(W_{T-1}-S_{T-1}\right)\sigma_\varepsilon\left[\frac{\theta\left(W_{T-1}-S_{T-1}\right)}{\sigma_\varepsilon}+\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})}{\sigma_\varepsilon}\right)}\right]\right\}$$

$$=\min_{\{S_{T-1}\}}\left[S_{T-1}\sigma_\varepsilon\psi\left(\xi S_{T-1}\right)+\left(W_{T-1}-S_{T-1}\right)\sigma_\varepsilon\psi\left\{\xi\left(W_{T-1}-S_{T-1}\right)\right\}\right]$$

We show this to be a convex function with a unique minimum. Let us start with,

$$G\left(x\right)=\frac{\phi\left(x\right)}{\Phi\left(x\right)}\quad |\ \forall x>0$$

$$\frac{\partial G\left(x\right)}{\partial x}=\frac{\phi'\left(x\right)}{\Phi\left(x\right)}-\left[\frac{\phi\left(x\right)}{\Phi\left(x\right)}\right]^2$$

$$\frac{\partial G\left(x\right)}{\partial x}=-\frac{x\phi\left(x\right)}{\Phi\left(x\right)}-\left[\frac{\phi\left(x\right)}{\Phi\left(x\right)}\right]^2$$

$$\left[\because\ \frac{\partial\phi\left(x\right)}{\partial\left(x\right)}=-x\phi\left(x\right)\ ;\ \frac{\partial\Phi\left(x\right)}{\partial\left(x\right)}=\phi\left(x\right)\right]$$

$$\frac{\partial G\left(x\right)}{\partial x}=\frac{\phi\left(x\right)}{\Phi\left(x\right)}\left[-x-\frac{\phi\left(x\right)}{\Phi\left(x\right)}\right]<0\quad |\ \forall x>0$$

$$\frac{\partial^2 G\left(x\right)}{\partial x^2} = -\frac{\phi\left(x\right)}{\Phi\left(x\right)}-x\left\{-\frac{x\phi\left(x\right)}{\Phi\left(x\right)}-\left[\frac{\phi\left(x\right)}{\Phi\left(x\right)}\right]^2\right\}-\frac{2\phi\left(x\right)\phi'\left(x\right)}{\Phi^2\left(x\right)}+\frac{2\phi^3\left(x\right)}{\Phi^3\left(x\right)}$$

$$= \frac{-\phi\left(x\right)\Phi^2\left(x\right)+x^2\phi\left(x\right)\Phi^2\left(x\right)+x\phi^2\left(x\right)\Phi\left(x\right)+2x\phi^2\left(x\right)\Phi\left(x\right)+2\phi^3\left(x\right)}{\Phi^3\left(x\right)}$$

$$= \frac{\phi\left(x\right)\left\{-\Phi^2\left(x\right)+x^2\Phi^2\left(x\right)+3x\phi\left(x\right)\Phi\left(x\right)+2\phi^2\left(x\right)\right\}}{\Phi^3\left(x\right)}$$

Consider the following, $\forall x > 0$

$$\text{Let } K(x) = 3x\phi(x)\Phi(x) + 2\phi^2(x) + \left(x^2 - 1\right)\Phi^2(x)$$

For $x \geq 1, K(x) > 0$. Also,

$$K(0) = \frac{1}{\pi} - \frac{1}{4} > 0$$

$$\left[\because \ \phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \ ; \ \phi(0) = \frac{1}{\sqrt{2\pi}} \ ; \ \Phi(x) = \frac{1}{2}\left\{1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right\} \ ; \ \Phi(0) = \frac{1}{2}\right]$$

For $x \in (0, 1)$,

$$K'(x) = 3\phi(x)\Phi(x) - 3x^2\phi(x)\Phi(x) + 3x\phi^2(x) - 4x\phi^2(x) + \left(x^2 - 1\right)2\Phi(x)\phi(x) + 2x\Phi^2(x)$$

$$K'(x) = x\left\{2\Phi^2(x) - \phi^2(x)\right\} + \left\{1 - x^2\right\}\phi(x)\Phi(x) \geq xL(x)$$

where, $L(x) = \left\{2\Phi^2(x) - \phi^2(x)\right\}$. Further, $\Phi(x) \geq \frac{1}{2}$ and $\phi^2(x) \leq \frac{1}{2\pi} \Rightarrow L(x) \geq \frac{1}{2} - \frac{1}{2\pi} > 0$. $K'(x) > 0 \Rightarrow K(x)$ is increasing. Hence, $K(x) > K(0) > 0 \ \mid \forall u \in (0, 1)$. This gives, $K(x) > 0$ and $\frac{\partial^2 G(x)}{\partial x^2} > 0 \mid \forall x \in (0, \infty)$. It is worth noting the following asymptotic properties

$$\left[\because \lim_{x \to 0+} \frac{\partial^2 G(x)}{\partial x^2} > 0 \ ; \ \lim_{x \to \infty} \frac{\partial^2 G(x)}{\partial x^2} = 0 \ ; \ \frac{\partial G(x)}{\partial x} < 0 \quad \mid \forall x > 0\right]$$

Next we show that $f(a - x)$ is convex, given $f''(x) > 0 \ ; \ x > 0$

$$\text{Let } y = a - x$$

$$\frac{\partial f(y)}{\partial x} = \frac{\partial f(y)}{\partial y}\frac{\partial(a - x)}{\partial x} \ ; \ a > x$$

$$= (-1) f'(y)$$

$$\frac{\partial^2 f(y)}{\partial x^2} = (-1)\frac{\partial f'(y)}{\partial y}\frac{\partial(a - x)}{\partial x}$$

$$= f''(y)$$

$$> 0 \ [\because \ f''(y) > 0 \ \mid \forall y > 0]$$

We can similarly show that $f(bx)$ is convex if $f(x)$ is convex (in our case, $b > 0$, but the result holds $\forall b$).

Next we derive conditions when $x^2 + xf(x)$ is convex, given $f''(x) > 0$; $x > 0$

$$\begin{aligned} \text{Let } g(x) &= x^2 + xf(x) \\ g'(x) &= 2x + xf'(x) + f(x) \\ g''(x) &= 2 + xf''(x) + 2f'(x) \end{aligned}$$

$$g''(x) > 0 \text{ if } f'(x) > 0 \text{ or if } \; 2 + xf''(x) > |2f'(x)|$$

Finally,

$$\text{Let } Q(x) = x^2 + x\frac{\phi(x)}{\Phi(x)}$$

$$\frac{\partial Q(x)}{\partial x} = 2x + x\left[-\frac{x\phi(x)}{\Phi(x)} - \left\{\frac{\phi(x)}{\Phi(x)}\right\}^2\right] + \frac{\phi(x)}{\Phi(x)}$$

$$\left.\frac{\partial Q(x)}{\partial x}\right|_{x=0} = \frac{\phi(0)}{\Phi(0)} > 0$$

$$\frac{\partial^2 Q(x)}{\partial x^2} = 2 + x\phi(x)\left[\frac{-\Phi^2(x) + x^2\Phi^2(x) + 3x\phi(x)\Phi(x) + 2\phi^2(x)}{\Phi^3(x)}\right] + 2\left[-x\frac{\phi(x)}{\Phi(x)} - \left\{\frac{\phi(x)}{\Phi(x)}\right\}^2\right]$$

$$= 2 + \phi(x)\left[\frac{x^3\Phi^2(x) + 3x^2\phi(x)\Phi(x) + 2x\phi^2(x) - 3x\Phi^2(x) - 2\phi(x)\Phi(x)}{\Phi^3(x)}\right]$$

$$= \left[\frac{2\Phi^3(x) + x^3\Phi^2(x)\phi(x) + 3x^2\phi^2(x)\Phi(x) + 2x\phi^3(x) - 3x\Phi^2(x)\phi(x) - 2\phi^2(x)\Phi(x)}{\Phi^3(x)}\right]$$

$$\text{Let, } K(x) = 2\Phi^3(x) + 2x\phi^3(x) + x^3\Phi^2(x)\phi(x) + 3x^2\phi^2(x)\Phi(x) - 3x\Phi^2(x)\phi(x) - 2\phi^2(x)\Phi(x)$$

We need to show that $K(x) > 0 \; |\forall x > 0$. First we note that,

$$K(0) = \frac{1}{4} - \frac{1}{2\pi} > 0$$

We can write this as,

$$K(x) = 2x\phi^3(x) + x^3\Phi^2(x)\phi(x) + 3x^2\phi^2(x)\Phi(x) + \Phi(x)\left[2\Phi^2(x) - 3x\Phi(x)\phi(x) - 2\phi^2(x)\right]$$

We then need to show,

$$L(x) = \left[2\Phi^2(x) - 3x\Phi(x)\phi(x) - 2\phi^2(x)\right] > 0 \,|\forall x > 0$$

$$L(0) = \left[2\Phi^2(0) - 2\phi^2(0)\right] = \frac{1}{2} - \frac{1}{\pi} > 0$$

$$\begin{aligned}\frac{\partial L(x)}{\partial x} &= 4\Phi(x)\phi(x) - 3\Phi(x)\phi(x) - 3x\phi^2(x) + 3x^2\Phi(x)\phi(x) + 4x\phi^2(x)\\ &= \Phi(x)\phi(x) + 3x^2\Phi(x)\phi(x) + x\phi^2(x) > 0 \,|\forall x \geq 0\end{aligned}$$

Therefore $L(x)$ is an increasing function on the interval $[0, \infty)$. Its minimum must be at $L(0) > 0$, proving $L(x) > 0$ and $\frac{\partial^2 Q(x)}{\partial x^2} > 0 \;|\, \forall x \in (0, \infty)$. It is worth noting the following asymptotic properties and the graphical results shown in the main text,

$$\left[\because \lim_{x \to 0+} \frac{\partial^2 Q(x)}{\partial x^2} > 0\,;\; \lim_{x \to \infty} \frac{\partial^2 Q(x)}{\partial x^2} > 0\,;\; \frac{\partial Q(x)}{\partial x} > 0 \quad |\, \forall x > 0\right]$$

□

11.3 Proof of Proposition 2

Proof. Consider,

$$V_{T-1}(P_{T-2}, W_{T-1}) = \min_{\{S_{T-1}\}} \left[S_{T-1}\sigma_\varepsilon \psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon \psi\{\xi(W_{T-1} - S_{T-1})\}\right]$$

$$\text{Here, } \psi(u) = u + \phi(u)/\Phi(u)\;,\; \xi = \frac{\theta}{\sigma_\varepsilon},$$

First Order Conditions (FOC) give,

$$\frac{\partial}{\partial S_{T-1}}\left[S_{T-1}\sigma_\varepsilon \psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon \psi\{\xi(W_{T-1} - S_{T-1})\}\right] = 0$$

$$\xi S_{T-1}\psi'(\xi S_{T-1}) + \psi(\xi S_{T-1}) - \xi(W_{T-1} - S_{T-1})\psi'(\xi\{W_{T-1} - S_{T-1}\}) - \psi(\xi\{W_{T-1} - S_{T-1}\}) = 0$$

$$\begin{aligned}&\xi^2 S_{T-1}\left\{1 - \frac{\xi S_{T-1}\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} - \left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})}\right]^2\right\} + \left\{\xi S_{T-1} + \frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})}\right\}\\ &+\xi^2(W_{T-1} - S_{T-1})\left\{-1 + \frac{\xi(W_{T-1} - S_{T-1})\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} + \left[\frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})}\right]^2\right\}\\ &-\left\{\xi\{W_{T-1} - S_{T-1}\} + \frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})}\right\} = 0\end{aligned}$$

$$\left[\because \ \psi'(u) = 1 - \frac{u\phi(u)}{\Phi(u)} - \left[\frac{\phi(u)}{\Phi(u)}\right]^2 \text{ and } \psi(u) = u + \frac{\phi(u)}{\Phi(u)}\right]$$

$$S_{T-1} + \frac{1}{\xi}S_{T-1} + \frac{1}{\xi^2}\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + \frac{\xi(W_{T-1}-S_{T-1})^2\phi(\xi\{W_{T-1}-S_{T-1}\})}{\Phi(\xi\{W_{T-1}-S_{T-1}\})}$$
$$+(W_{T-1}-S_{T-1})\left[\frac{\phi(\xi\{W_{T-1}-S_{T-1}\})}{\Phi(\xi\{W_{T-1}-S_{T-1}\})}\right]^2 \quad =$$
$$(W_{T-1}-S_{T-1}) + \frac{1}{\xi}\{W_{T-1}-S_{T-1}\} + \frac{1}{\xi^2}\frac{\phi(\xi\{W_{T-1}-S_{T-1}\})}{\Phi(\xi\{W_{T-1}-S_{T-1}\})}$$
$$+\frac{\xi S_{T-1}^2\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + S_{T-1}\left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})}\right]^2$$

Setting $S_{T-1} = W_{T-1}/2$ gives $RHS = LHS$. We have,

$$\begin{aligned} V_{T-1}(P_{T-2}, W_{T-1}) &= S_{T-1}\sigma_\varepsilon\left[\frac{\theta S_{T-1}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}\right] \\ &+ (W_{T-1}-S_{T-1})\sigma_\varepsilon\left[\frac{\theta(W_{T-1}-S_{T-1})}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})}{\sigma_\varepsilon}\right)}\right] \end{aligned}$$

$$V_{T-1}(P_{T-2}, W_{T-1}) = W_{T-1}\sigma_\varepsilon\left[\frac{\theta W_{T-1}}{2\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_{T-1}}{2\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_{T-1}}{2\sigma_\varepsilon}\right)}\right]$$

Absent closed form solutions, numerical techniques using $\xi_1 > 0$ can be tried. We can also set $S_{T-1} \approx \omega_1(W_{T-1})$ using a well behaved (continuous and differentiable) function, ω_1. But the former approach is simpler and lends itself easily to numerical solutions that we will attempt in the more complex laws of motion to follow.

$$S_{T-1} \approx \xi_1 W_{T-1}$$

or with additional terms including non-linear regressions as,

$$S_{T-1} \approx \xi_0 + \xi_1 W_{T-1} + \xi_2(W_{T-1})^2 \text{ OR } S_{T-1} \approx \xi_0(W_{T-1})^{\xi_1}$$

$$V_{T-1}(P_{T-2}, W_{T-1}) = [\sigma_\varepsilon\xi_1 W_{T-1}\psi(\xi\xi_1 W_{T-1}) + \{W_{T-1} - \xi_1 W_{T-1}\}\sigma_\varepsilon\psi\{\xi(W_{T-1} - \xi_1 W_{T-1})\}]$$

$$= \sigma_\varepsilon W_{T-1}[\xi_1\psi(\xi\xi_1 W_{T-1}) + (1-\xi_1)\psi\{\xi W_{T-1}(1-\xi_1)\}]$$

$$= \psi_1(W_{T-1}) \text{ ,here, } \psi_1 \text{ is a convex function.}$$

Continuing the recursion,

$$V_{T-2}\left(P_{T-3},W_{T-2}\right)=\min_{\{S_{T-2}\}}E_{T-2}\left[\max\left\{\left(P_{T-2}-P_{T-3}\right),0\right\}S_{T-2}+V_{T-1}\left(P_{T-2},W_{T-1}\right)\right]$$

$$=\min_{\{S_{T-2}\}}E_{T-2}\left[\max\left\{\left(\theta S_{T-2}+\varepsilon_{T-2}\right),0\right\}S_{T-2}+V_{T-1}\left(P_{T-3}+\theta S_{T-2}+\varepsilon_{T-3},W_{T-2}-S_{T-2}\right)\right]$$

$$=\min_{\{S_{T-2}\}}\left[S_{T-2}\sigma_\varepsilon\left\{\frac{\theta S_{T-2}}{\sigma_\varepsilon}+\frac{\phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}\right\}+\left(W_{T-2}-S_{T-2}\right)\sigma_\varepsilon\left\{\frac{\theta\left(W_{T-2}-S_{T-2}\right)}{2\sigma_\varepsilon}+\frac{\phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}\right\}\right]$$

First Order Conditions (FOC) give,

$$3\theta S_{T-2}-\theta W_{T-2}+\frac{\sigma_\varepsilon\phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}-\frac{\theta^2S_{T-2}^2\phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}-\theta S_{T-2}\left[\frac{\phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}\right]^2$$

$$-\frac{\sigma_\varepsilon\phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}+\frac{\theta^2\left(W_{T-2}-S_{T-2}\right)^2}{4\sigma_\varepsilon}\frac{\phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}+\frac{\theta\left(W_{T-2}-S_{T-2}\right)}{2}\left\{\frac{\phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2}-S_{T-2})}{2\sigma_\varepsilon}\right]}\right\}^2=0$$

This gives, $S_{T-2}=W_{T-2}/3$ and the corresponding value function as,

$$V_{T-2}\left(P_{T-3},W_{T-2}\right)=\frac{W_{T-2}}{3}\sigma_\varepsilon\left\{\frac{\theta}{\sigma_\varepsilon}\frac{W_{T-2}}{3}+\frac{\phi\left(\frac{\theta}{\sigma_\varepsilon}\frac{W_{T-2}}{3}\right)}{\Phi\left(\frac{\theta}{\sigma_\varepsilon}\frac{W_{T-2}}{3}\right)}\right\}+\left(\frac{2W_{T-2}}{3}\right)\sigma_\varepsilon\left\{\frac{\theta}{2\sigma_\varepsilon}\frac{2W_{T-2}}{3}+\frac{\phi\left[\frac{\theta}{2\sigma_\varepsilon}\frac{2W_{T-2}}{3}\right]}{\Phi\left[\frac{\theta}{2\sigma_\varepsilon}\frac{2W_{T-2}}{3}\right]}\right\}$$

$$V_{T-2}\left(P_{T-3},W_{T-2}\right)=\sigma_\varepsilon W_{T-2}\left[\frac{\theta W_{T-2}}{3\sigma_\varepsilon}+\frac{\phi\left(\frac{\theta W_{T-2}}{3\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_{T-2}}{3\sigma_\varepsilon}\right)}\right]$$

We show the general case using induction. Let the value function hold for $T-K$

$$V_{T-K}\left(P_{T-K-1},W_{T-K}\right)=\sigma_\varepsilon W_{T-K}\left[\frac{\theta W_{T-K}}{\left(K+1\right)\sigma_\varepsilon}+\frac{\phi\left(\frac{\theta W_{T-K}}{(K+1)\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_{T-K}}{(K+1)\sigma_\varepsilon}\right)}\right]$$

Continuing the recursion,

$$\begin{aligned}V_{T-K-1}\left(P_{T-K-2},W_{T-K-1}\right)\quad=\quad&\min_{\{S_{T-K-1}\}}E_{T-K-1}\left[\max\left\{\left(P_{T-K-1}-P_{T-K-2}\right),0\right\}S_{T-K-1}\right.\\&\left.+V_{T-K}\left(P_{T-K-1},W_{T-K}\right)\right]\end{aligned}$$

$$\begin{aligned}=\quad&\min_{\{S_{T-K-1}\}}E_{T-K-1}\left[\max\left\{\left(\theta S_{T-K-1}+\varepsilon_{T-K-1}\right),0\right\}S_{T-K-1}\right.\\&\left.+V_{T-K}\left(P_{T-K-2}+\theta S_{T-K-1}+\varepsilon_{T-K-2},W_{T-K-1}-S_{T-K-1}\right)\right]\end{aligned}$$

$$= \min_{\{S_{T-K-1}\}} \left[S_{T-K-1}\sigma_\varepsilon \left\{ \frac{\theta S_{T-K-1}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-K-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-K-1}}{\sigma_\varepsilon}\right)} \right\} \right.$$

$$\left. + \left(W_{T-K-1} - S_{T-K-1}\right)\sigma_\varepsilon \left\{ \frac{\theta\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)\sigma_\varepsilon} + \frac{\phi\left[\frac{\theta(W_{T-K-1}-S_{T-K-1})}{(K+1)\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-K-1}-S_{T-K-1})}{(K+1)\sigma_\varepsilon}\right]} \right\} \right]$$

$$= \min_{\{S_{T-K-1}\}} \left[S_{T-K-1} \left\{ \xi S_{T-K-1} + \frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} \right\} \right.$$

$$\left. + \left(W_{T-K-1} - S_{T-K-1}\right) \left\{ \frac{\xi\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)} + \frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]} \right\} \right]$$

First Order Conditions (FOC) give,

$$\left\{ \xi S_{T-K-1} + \frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} \right\}$$

$$+ S_{T-K-1} \left\{ \xi - \xi^2 S_{T-K-1} \frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} - \xi \left[\frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} \right]^2 \right\}$$

$$- \left\{ \frac{\xi\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)} + \frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]} \right\}$$

$$+ \left(W_{T-K-1} - S_{T-K-1}\right) \left\{ -\frac{\xi}{(K+1)} + \frac{\xi^2\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)^2} \frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]} \right.$$

$$\left. + \frac{\xi}{(K+1)} \left[\frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]} \right]^2 \right\} = 0$$

$$2\xi S_{T-K-1} + \frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} + \frac{\xi^2\left(W_{T-K-1} - S_{T-K-1}\right)^2}{(K+1)^2} \frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}$$

$$+ \frac{\xi\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)} \left[\frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]} \right]^2 =$$

$$\xi^2 S_{T-K-1}^2 \frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} + \xi S_{T-K-1} \left[\frac{\phi\left(\xi S_{T-K-1}\right)}{\Phi\left(\xi S_{T-K-1}\right)} \right]^2$$

$$+ \frac{2\xi\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)} + \frac{\phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}{\Phi\left[\frac{\xi(W_{T-K-1}-S_{T-K-1})}{(K+1)}\right]}$$

This gives, $S_{T-K-1} = W_{T-K-1}/(K+2)$ and the corresponding value function is,

$$\begin{aligned} V_{T-K-1}\left(P_{T-K-2}, W_{T-K-1}\right) \quad = \quad & \frac{W_{T-K-1}}{(K+2)}\sigma_{\varepsilon}\left\{\frac{\theta}{\sigma_{\varepsilon}}\frac{W_{T-K-1}}{(K+2)} + \frac{\phi\left(\frac{\theta}{\sigma_{\varepsilon}}\frac{W_{T-K-1}}{(K+2)}\right)}{\Phi\left(\frac{\theta}{\sigma_{\varepsilon}}\frac{W_{T-K-1}}{(K+2)}\right)}\right\} \\ & + \left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)}\right)\sigma_{\varepsilon}\left\{\frac{\theta}{(K+1)\sigma_{\varepsilon}}\left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)}\right)\right. \\ & \left. + \frac{\phi\left[\frac{\theta}{(K+1)\sigma_{\varepsilon}}\left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)}\right)\right]}{\Phi\left[\frac{\theta}{(K+1)\sigma_{\varepsilon}}\left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)}\right)\right]}\right\} \end{aligned}$$

$$V_{T-K-1}\left(P_{T-K-2}, W_{T-K-1}\right) = \sigma_{\varepsilon} W_{T-K-1}\left[\frac{\theta}{\sigma_{\varepsilon}}\frac{W_{T-K-1}}{(K+2)} + \frac{\phi\left(\frac{\theta}{\sigma_{\varepsilon}}\frac{W_{T-K-1}}{(K+2)}\right)}{\Phi\left(\frac{\theta}{\sigma_{\varepsilon}}\frac{W_{T-K-1}}{(K+2)}\right)}\right]$$

This completes the induction.

□

11.4 Proof of Proposition 3

Proof. Consider,

$$V_T\left(P_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T + \varepsilon_T\right), 0\right\} W_T\right]$$

$$V_T\left(P_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T^2 + W_T\varepsilon_T\right), 0\right\}\right]$$

$$V_T\left(P_{T-1}, W_T\right) = E_T\left[\left(\theta W_T^2 + W_T\varepsilon_T\right)\middle|\left(\theta W_T^2 + W_T\varepsilon_T\right) > 0\right]$$

$$\{\because\ E\left[max\left(X, c\right)\right] = E\left[X\,|X > c\right] Pr\left[X > c\right] + E\left[c\,|X{\leq}c\right] Pr\left[X{\leq}c\right]\ \}$$

This is of the form, $E\left[Y|\,Y > 0\right]$ where, $Y = \left(\theta W_T^2 + W_T\varepsilon_T\right)$. We then need to calculate,

$$E_T\left[\left(\theta W_T^2 + W_T\sigma_{\varepsilon}Z\right)\middle|\,Z > \left(-\frac{\theta W_T}{\sigma_{\varepsilon}}\right)\right] \quad , where\ Z \sim N\left(0, 1\right)$$

$$\left[\because\ Y \sim N\left(\theta W_T^2, W_T^2\sigma_{\varepsilon}^2\right) \equiv Y \sim N\left(\mu, \sigma^2\right) \Rightarrow Y = \mu + \sigma Z\ ;\ Y > 0 \Rightarrow Z > -\mu/\sigma\right]$$

We have for every standard normal distribution, Z, and for every u, $Pr\left[Z > -u\right] = Pr\left[Z < u\right] = \mathbf{\Phi}\left(u\right)$. Here, ϕ and $\mathbf{\Phi}$ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E\left[Z \middle| Z > -u\right] &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[\int_{-u}^{\infty} t\phi\left(t\right)dt\right] \\ &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[-\phi\left(t\right)\Big|_{-u}^{\infty}\right] = \frac{\phi\left(u\right)}{\mathbf{\Phi}\left(u\right)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E\left[Y \middle| Y > 0\right] &= \mu + \sigma E\left[Z \middle| Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma\phi\left(\mu/\sigma\right)}{\mathbf{\Phi}\left(\mu/\sigma\right)} \end{aligned}$$

Setting, $\psi\left(u\right) = u + \phi\left(u\right)/\Phi\left(u\right)$,

$$E\left[Y \middle| Y > 0\right] = \sigma\psi\left(\mu/\sigma\right)$$

$$\begin{aligned} V_T\left(P_{T-1}, W_T\right) &= E_T\left[\left(\theta W_T^2 + W_T\varepsilon_T\right)\middle|\left(\theta W_T^2 + W_T\varepsilon_T\right) > 0\right] \\ &= W_T\sigma_\varepsilon\left[\frac{\theta W_T}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)}\right] = \sigma_\varepsilon W_T\psi\left(\xi W_T\right)\ ,\ \xi = \frac{\theta}{\sigma_\varepsilon} \end{aligned}$$

In the next to last period, $T-1$, the Bellman equation is,

$$V_{T-1}\left(P_{T-2}, W_{T-1}\right) = \min_{\{S_{T-1}\}} E_{T-1}\left[\max\left\{\left(P_{T-1} - P_{T-2}\right), 0\right\}W_{T-1} + V_T\left(P_{T-1}, W_T\right)\right]$$

$$= \min_{\{S_{T-1}\}} E_{T-1}\left[\max\left\{\left(\theta S_{T-1} + \varepsilon_{T-1}\right), 0\right\}W_{T-1} + V_T\left(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1}, W_{T-1} - S_{T-1}\right)\right]$$

$$= \min_{\{S_{T-1}\}}\left\{W_{T-1}\sigma_\varepsilon\left[\frac{\theta S_{T-1}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}\right] + \left(W_{T-1} - S_{T-1}\right)\sigma_\varepsilon\left[\frac{\theta\left(W_{T-1} - S_{T-1}\right)}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta\left(W_{T-1} - S_{T-1}\right)}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta\left(W_{T-1} - S_{T-1}\right)}{\sigma_\varepsilon}\right)}\right]\right\}$$

$$V_{T-1}\left(P_{T-2}, W_{T-1}\right) = \min_{\{S_{T-1}\}}\left[W_{T-1}\sigma_\varepsilon\psi\left(\xi S_{T-1}\right) + \left(W_{T-1} - S_{T-1}\right)\sigma_\varepsilon\psi\left\{\xi\left(W_{T-1} - S_{T-1}\right)\right\}\right]$$

$$\text{Here, } \psi\left(u\right) = u + \phi\left(u\right)/\Phi\left(u\right)\ ;\ \xi = \frac{\theta}{\sigma_\varepsilon}\ ;\ \text{Note that, } W_{T-1} = S_{T-1} + W_T$$

We can show the above expression to be a convex function with a unique minimum, since it is the sum of the portion shown to be convex earlier (Proposition 1, Appendix 11.2), another convex function and a linear component.

First Order Conditions (FOC) give,

$$\frac{\partial}{\partial S_{T-1}}\left[W_{T-1}\sigma_{\varepsilon}\psi\left(\xi S_{T-1}\right)+\left(W_{T-1}-S_{T-1}\right)\sigma_{\varepsilon}\psi\left\{\xi\left(W_{T-1}-S_{T-1}\right)\right\}\right]=0$$

$$\xi W_{T-1}\psi'\left(\xi S_{T-1}\right)-\xi\left(W_{T-1}-S_{T-1}\right)\psi'\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)-\psi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)=0$$

$$\begin{aligned}\left(\xi W_{T-1}\right)\left\{1-\frac{\xi S_{T-1}\phi\left(\xi S_{T-1}\right)}{\Phi\left(\xi S_{T-1}\right)}-\left[\frac{\phi\left(\xi S_{T-1}\right)}{\Phi\left(\xi S_{T-1}\right)}\right]^{2}\right\}\\ -\xi\left(W_{T-1}-S_{T-1}\right)\left\{1-\frac{\xi\left(W_{T-1}-S_{T-1}\right)\phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}{\Phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}-\left[\frac{\phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}{\Phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}\right]^{2}\right\}\\ -\left\{\xi\left\{W_{T-1}-S_{T-1}\right\}+\frac{\phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}{\Phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}\right\} &= 0\end{aligned}$$

$$\left[\because\ \psi'\left(u\right)=1-\frac{u\phi\left(u\right)}{\Phi\left(u\right)}-\left[\frac{\phi\left(u\right)}{\Phi\left(u\right)}\right]^{2}\text{ and }\psi\left(u\right)=u+\frac{\phi\left(u\right)}{\Phi\left(u\right)}\right]$$

$$\begin{aligned}W_{T-1}+\frac{\xi\left(W_{T-1}-S_{T-1}\right)^{2}\phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}{\Phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}+\left(W_{T-1}-S_{T-1}\right)\left[\frac{\phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}{\Phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}\right]^{2} &=\\ \left(W_{T-1}-S_{T-1}\right)+\left\{W_{T-1}-S_{T-1}\right\}+\frac{1}{\xi}\frac{\phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}{\Phi\left(\xi\left\{W_{T-1}-S_{T-1}\right\}\right)}+\frac{\xi W_{T-1}S_{T-1}\phi\left(\xi S_{T-1}\right)}{\Phi\left(\xi S_{T-1}\right)}+W_{T-1}\left[\frac{\phi\left(\xi S_{T-1}\right)}{\Phi\left(\xi S_{T-1}\right)}\right]^{2}\end{aligned}$$

□

11.5 Proof of Proposition 4

Proof. Consider,

$$V_T\left(P_{T-1},X_{T-1},W_T\right)=E_T\left[\max\left\{\left(\theta W_T+\varepsilon_T+\gamma X_T\right),0\right\}W_T\right]$$

$$V_T\left(P_{T-1},X_{T-1},W_T\right)=E_T\left[\max\left\{\left(\theta W_T^2+W_T\varepsilon_T+\gamma\rho W_T X_{T-1}+\gamma W_T\eta_T\right),0\right\}\right]$$

$$=E_T\left[\left(\theta W_T^2+W_T\varepsilon_T+\gamma\rho W_T X_{T-1}+\gamma W_T\eta_T\right)\middle|\left(\theta W_T^2+W_T\varepsilon_T+\gamma\rho W_T X_{T-1}+\gamma W_T\eta_T\right)>0\right]$$

$$\left\{\because\ E\left[max\left(X,c\right)\right]=E\left[X\left|X>c\right.\right]Pr\left[X>c\right]+E\left[c\left|X\leq c\right.\right]Pr\left[X\leq c\right]\ \right\}$$

This is of the form, $E\left[Y|\,Y>0\right]$ where, $Y=\left(\theta W_T^2+W_T\varepsilon_T+\gamma\rho W_T X_{T-1}+\gamma W_T\eta_T\right)$. We then need to calculate,

$$E_T\left[\left\{\theta W_T^2+\gamma\rho W_T X_{T-1}+W_T\left(\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)Z\right\}\middle|\,Z>\left(-\frac{\theta W_T+\gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)\right]\quad ,where\ Z\sim N\left(0,1\right)$$

$$\left[\because\ X \sim N(\mu_X, \sigma_X^2)\ ;\ Y \sim N(\mu_Y, \sigma_Y^2)\ ;\ U = X + Y \Rightarrow U \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)\right]$$

$$\left[\because\ Y \sim N\left(\theta W_T^2 + \gamma\rho W_T X_{T-1}, W_T^2\left\{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2\right\}\right) \equiv Y \sim N\left(\mu, \sigma^2\right) \Rightarrow Y = \mu + \sigma Z\ ;\ Y > 0 \Rightarrow Z > -\mu/\sigma\right]$$

We have for every standard normal distribution, Z, and for every u, $Pr\left[Z > -u\right] = Pr\left[Z < u\right] = \mathbf{\Phi}\left(u\right)$. Here, ϕ and $\mathbf{\Phi}$ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E\left[Z \middle| Z > -u\right] &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[\int_{-u}^{\infty} t\phi\left(t\right)dt\right] \\ &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[-\phi\left(t\right)\big|_{-u}^{\infty}\right] = \frac{\phi\left(u\right)}{\mathbf{\Phi}\left(u\right)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E\left[Y \middle| Y > 0\right] &= \mu + \sigma E\left[Z \middle| Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma\phi\left(\mu/\sigma\right)}{\mathbf{\Phi}\left(\mu/\sigma\right)} \end{aligned}$$

Setting, $\psi\left(u\right) = u + \phi\left(u\right)/\Phi\left(u\right)$,

$$E\left[Y \middle| Y > 0\right] = \sigma\psi\left(\mu/\sigma\right)$$

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right)\middle|\left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right) > 0\right]$$

$$= W_T\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\left[\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]$$

$$= \left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)W_T\psi\left(\xi W_T\right)\ ,\ \xi W_T = \frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}$$

In the next to last period, $T-1$, the Bellman equation is,

$$V_{T-1}\left(P_{T-2}, X_{T-2}, W_{T-1}\right) = \min_{\{S_{T-1}\}} E_{T-1}\left[\max\left\{\left(P_{T-1} - P_{T-2}\right), 0\right\}S_{T-1} + V_T\left(P_{T-1}, X_{T-1}, W_T\right)\right]$$

$$\begin{aligned} = \min_{\{S_{T-1}\}} & E_{T-1}\left[\max\left\{\left(\theta S_{T-1}^2 + S_{T-1}\varepsilon_{T-1} + \gamma\rho S_{T-1}X_{T-2} + \gamma S_{T-1}\eta_{T-1}\right), 0\right\}\right. \\ & + \left. V_T\left(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1} + \gamma\rho X_{T-2} + \gamma\eta_{T-1}, \rho X_{T-2} + \eta_{T-1}, W_{T-1} - S_{T-1}\right)\right] \end{aligned}$$

$$= \min_{\{S_{T-1}\}} \left\{ S_{T-1}\left(\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)\left[\frac{\theta S_{T-1}+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta S_{T-1}+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta S_{T-1}+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}\right]\right.$$

$$\left. + \{W_{T-1}-S_{T-1}\}\left(\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)\left[\frac{\theta(W_{T-1}-S_{T-1})+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}\right]\right\}$$

$$= \min_{\{S_{T-1}\}}\left[S_{T-1}\left(\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)\psi(\xi_1 S_{T-1}) + (W_{T-1}-S_{T-1})\left(\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)\psi\{\xi_1(W_{T-1}-S_{T-1})\}\right]$$

$$\text{Here, } \xi_1 S_{T-1} = \frac{\theta S_{T-1}+\gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}} \text{ Also, let } \alpha_1 = \gamma\rho X_{T-2},\ \beta = \sqrt{\gamma^2\sigma_\eta^2+\sigma_\varepsilon^2}$$

$$= \min_{\{S_{T-1}\}}\left\{\left[\theta S_{T-1}^2 + \alpha_1 S_{T-1} + \beta S_{T-1}\frac{\phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}\right]\right.$$

$$\left. + \left[\theta(W_{T-1}-S_{T-1})^2 + \alpha_1(W_{T-1}-S_{T-1}) + \beta(W_{T-1}-S_{T-1})\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}\right]\right\}$$

This is a convex function and numerical solutions can be obtained at each stage of the recursion or taking First Order Conditions give,

$$2\theta S_{T-1} + \alpha_1 + \beta\left\{\frac{\phi\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)} + \frac{\theta S_{T-1}}{\beta}\left[-\frac{\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)\phi\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)} - \left\{\frac{\phi\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha_1}{\beta}\right)}\right\}^2\right]\right\}$$

$$-2\theta(W_{T-1}-S_{T-1}) - \alpha_1 + \beta\left\{-\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}\right.$$

$$\left. + \frac{\theta(W_{T-1}-S_{T-1})}{\beta}\left[\frac{\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)} + \left\{\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha_1}{\beta}\right)}\right\}^2\right]\right\} = 0$$

$S_{T-1} = W_{T-1}/2$ solves this, giving the value function,

$$V_{T-1}(P_{T-2}, X_{T-2}, W_{T-1}) = \frac{\theta}{2}W_{T-1}^2 + \alpha_1 W_{T-1} + \beta W_{T-1}\frac{\phi\left(\frac{\theta W_{T-1}+2\alpha_1}{2\beta}\right)}{\Phi\left(\frac{\theta W_{T-1}+2\alpha_1}{2\beta}\right)}$$

Continuing the recursion,

$$V_{T-2}(P_{T-3}, X_{T-3}, W_{T-2}) = \min_{\{S_{T-2}\}} E_{T-2}\left[\max\{(P_{T-2}-P_{T-3}), 0\} S_{T-2} + V_{T-1}(P_{T-2}, X_{T-2}, W_{T-1})\right]$$

$$= \min_{\{S_{T-2}\}} E_{T-2}\left[\max\left\{\left(\theta S_{T-2}^2 + S_{T-2}\varepsilon_{T-2} + \gamma\rho S_{T-2}X_{T-3} + \gamma S_{T-2}\eta_{T-2}\right), 0\right\}\right.$$
$$\left. + \quad V_{T-1}\left(P_{T-3} + \theta S_{T-2} + \varepsilon_{T-2} + \gamma\rho X_{T-3} + \gamma\eta_{T-2}, \rho X_{T-3} + \eta_{T-2}, W_{T-2} - S_{T-2}\right)\right]$$

$$= \min_{\{S_{T-2}\}} \left\{ S_{T-2}\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\left[\frac{\theta S_{T-2} + \gamma\rho X_{T-3}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta S_{T-2} + \gamma\rho X_{T-3}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta S_{T-2} + \gamma\rho X_{T-3}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]\right.$$
$$\left. + \{W_{T-2} - S_{T-2}\}\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\left[\frac{\theta\{W_{T-2} - S_{T-2}\} + 2\gamma\rho X_{T-3}}{2\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta\{W_{T-2} - S_{T-2}\} + 2\gamma\rho X_{T-3}}{2\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta\{W_{T-2} - S_{T-2}\} + 2\gamma\rho X_{T-3}}{2\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]\right\}$$

$$= \min_{\{S_{T-2}\}} \left[S_{T-2}\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\psi\left(\xi_2 S_{T-2}\right) + \left(W_{T-2} - S_{T-2}\right)\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\psi\left\{\xi_2\left(W_{T-2} - S_{T-2}\right)\right\}\right]$$

$$\text{Here, } \xi_2 S_{T-2} = \frac{\theta S_{T-2} + 2\gamma\rho X_{T-3}}{2\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} \quad \text{Also, let } \alpha_2 = \gamma\rho X_{T-3},\ \beta = \sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}$$

$$= \min_{\{S_{T-2}\}} \left\{\left[\theta S_{T-2}^2 + \alpha_2 S_{T-2} + \beta S_{T-2}\frac{\phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)}\right]\right.$$
$$\left. + \left[\frac{\theta}{2}\left(W_{T-2} - S_{T-2}\right)^2 + \alpha_2\left(W_{T-2} - S_{T-2}\right) + \beta\left(W_{T-2} - S_{T-2}\right)\frac{\phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)}{\Phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)}\right]\right\}$$

This is a convex function and numerical solutions can be obtained at each stage of the recursion or taking First Order Conditions give,

$$2\theta S_{T-2} + \alpha_2 + \beta\left\{\frac{\phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)} + \frac{\theta S_{T-2}}{\beta}\left[-\frac{\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)\phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)} - \left\{\frac{\phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-2} + \alpha_2}{\beta}\right)}\right\}^2\right]\right\}$$
$$-\theta\left(W_{T-2} - S_{T-2}\right) - \alpha_2 + \beta\left\{-\frac{\phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)}{\Phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha}{2\beta}\right)}\right.$$
$$\left. + \frac{\theta\left(W_{T-2} - S_{T-2}\right)}{2\beta}\left[\frac{\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)\phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)}{\Phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)} + \left\{\frac{\phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)}{\Phi\left(\frac{\theta(W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta}\right)}\right\}^2\right]\right\} = 0$$

$S_{T-1} = W_{T-1}/3$ solves this, giving the value function,

$$V_{T-2}\left(P_{T-3}, X_{T-3}, W_{T-2}\right) = \frac{\theta}{3}W_{T-2}^2 + \alpha_2 W_{T-2} + \beta W_{T-2}\frac{\phi\left(\frac{\theta W_{T-2} + 3\alpha_2}{3\beta}\right)}{\Phi\left(\frac{\theta W_{T-2} + 3\alpha_2}{3\beta}\right)}$$

We show the general case using induction. Let the value function hold for $T-K$

$$V_{T-K}\left(P_{T-K-1},X_{T-K-1},W_{T-K}\right) = \frac{\theta}{(K+1)}W_{T-K}^{2}+\alpha_{K}W_{T-K}+\beta W_{T-K}\frac{\phi\left(\frac{\theta W_{T-K}+(K+1)\alpha_{K}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta W_{T-K}+(K+1)\alpha_{K}}{(K+1)\beta}\right)}$$

$$\begin{aligned}V_{T-K-1}\left(P_{T-K-2},X_{T-K-2},W_{T-K-1}\right) &= \min_{\{S_{T-K-1}\}} E_{T-K-1}\left[\max\left\{\left(P_{T-K-1}-P_{T-K-2}\right),0\right\}S_{T-K-1}\right.\\ &\quad \left.+V_{T-K}\left(P_{T-K-1},X_{T-K-1},W_{T-K}\right)\right]\end{aligned}$$

$$= \min_{\{S_{T-K-1}\}} E_{T-K-1}\left[\max\left\{\left(\theta S_{T-K-1}^{2}+S_{T-K-1}\varepsilon_{T-K-1}+\gamma\rho S_{T-K-1}X_{T-K-2}+\gamma S_{T-K-1}\eta_{T-K-1}\right),0\right\}\right.$$

$$\left.+V_{T-K}\left(P_{T-K-2}+\theta S_{T-K-1}+\varepsilon_{T-K-1}+\gamma\rho X_{T-K-2}+\gamma\eta_{T-K-1},\rho X_{T-K-2}+\eta_{T-K-1},W_{T-K-1}-S_{T-K-1}\right)\right]$$

$$\begin{aligned}= \min_{\{S_{T-K-1}\}}&\left\{S_{T-K-1}\left(\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}\right)\left[\frac{\theta S_{T-K-1}+\gamma\rho X_{T-K-2}}{\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}+\frac{\phi\left(\frac{\theta S_{T-K-1}+\gamma\rho X_{T-K-2}}{\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}\right)}{\Phi\left(\frac{\theta S_{T-K-1}+\gamma\rho X_{T-K-2}}{\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}\right)}\right]\right.\\ &+\left\{W_{T-K-1}-S_{T-K-1}\right\}\left(\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}\right)\\ &\left.\left[\frac{\theta\left\{W_{T-K-1}-S_{T-K-1}\right\}+(K+1)\gamma\rho X_{T-3}}{(K+1)\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}+\frac{\phi\left(\frac{\theta\{W_{T-K-1}-S_{T-K-1}\}+(K+1)\gamma\rho X_{T-K-2}}{(K+1)\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}\right)}{\Phi\left(\frac{\theta\{W_{T-K-1}-S_{T-K-1}\}+(K+1)\gamma\rho X_{T-K-2}}{(K+1)\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}\right)}\right]\right\}\end{aligned}$$

$$\begin{aligned}= \min_{\{S_{T-K-1}\}}&\left[S_{T-K-1}\left(\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}\right)\psi\left(\xi_{K+1}S_{T-K-1}\right)\right.\\ &\left.+\left(W_{T-K-1}-S_{T-K-1}\right)\left(\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}\right)\psi\left\{\xi_{K+1}\left(W_{T-K-1}-S_{T-K-1}\right)\right\}\right]\end{aligned}$$

Here, $\xi_{K+1}S_{T-K-1}=\dfrac{\theta S_{T-K-1}+2\gamma\rho X_{T-K-2}}{2\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}}$ Also, let $\alpha_{K+1}=\gamma\rho X_{T-K-2}$, $\beta=\sqrt{\gamma^{2}\sigma_{\eta}^{2}+\sigma_{\varepsilon}^{2}}$

$$= \min_{\{S_{T-K-1}\}} \left\{ \left[\theta S_{T-K-1}^2 + \alpha_{K+1} S_{T-K-1} + \beta S_{T-K-1} \frac{\phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)} \right] \right.$$

$$+ \left[\frac{\theta}{(K+1)} \left(W_{T-K-1} - S_{T-K-1}\right)^2 + \alpha_{K+1} \left(W_{T-K-1} - S_{T-K-1}\right) \right]$$

$$\left. + \left[\beta \left(W_{T-K-1} - S_{T-K-1}\right) \frac{\phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right] \right\}$$

This is a convex function and numerical solutions can be obtained at each stage of the recursion or taking First Order Conditions give,

$$2\theta S_{T-K-1} + \alpha_{K+1} + \beta \left\{ \frac{\phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)} \right.$$

$$\left. + \frac{\theta S_{T-K-1}}{\beta} \left[-\frac{\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)\phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)} - \left\{ \frac{\phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1}+\alpha_{K+1}}{\beta}\right)} \right\}^2 \right] \right\}$$

$$- \frac{2\theta}{(K+1)} \left(W_{T-K-1} - S_{T-K-1}\right) - \alpha_{K+1} + \beta \left\{ - \frac{\phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right.$$

$$+ \frac{\theta\left(W_{T-K-1} - S_{T-K-1}\right)}{(K+1)\beta} \left[\frac{\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)\phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right.$$

$$\left. \left. + \left\{ \frac{\phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1}-S_{T-K-1})+(K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right\}^2 \right] \right\} = 0$$

$S_{T-K-1} = W_{T-K-1} / (K+2)$ solves this, giving the value function and completing the induction.

$$V_{T-K-1}\left(P_{T-K-2}, X_{T-K-2}, W_{T-K-1}\right) = \frac{\theta}{(K+2)} W_{T-K-1}^2 + \alpha_{K+1} W_{T-K-1} + \beta W_{T-K-1} \frac{\phi\left(\frac{\theta W_{T-K-1}+(K+2)\alpha_{K+1}}{(K+2)\beta}\right)}{\Phi\left(\frac{\theta W_{T-K-1}+(K+2)\alpha_{K+1}}{(K+2)\beta}\right)}$$

□

11.6 Proof of Proposition 5

Proof. Consider,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T + \varepsilon_T + \gamma X_T\right), 0\right\} W_T\right]$$

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T\right), 0\right\}\right]$$

$$= E_T\left[\left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right)\middle|\left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right) > 0\right]$$

$$\{\because\ E\left[max\left(X,c\right)\right] = E\left[X\left|X > c\right.\right]Pr\left[X > c\right] + E\left[c\left|X \leq c\right.\right]Pr\left[X \leq c\right]\ \}$$

This is of the form, $E\left[Y|\,Y > 0\right]$ where, $Y = \left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right)$. We then need to calculate,

$$E_T\left[\left\{\theta W_T^2 + \gamma\rho W_T X_{T-1} + W_T\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)Z\right\}\middle| Z > \left(-\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)\right] \quad , where\ Z \sim N\left(0,1\right)$$

$$\left[\because\ X \sim N(\mu_X, \sigma_X^2)\ ;\ Y \sim N(\mu_Y, \sigma_Y^2)\ ;\ U = X + Y \Rightarrow U \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)\right]$$

$$\left[\because\ Y \sim N\left(\theta W_T^2 + \gamma\rho W_T X_{T-1}, W_T^2\left\{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2\right\}\right) \equiv Y \sim N\left(\mu, \sigma^2\right) \Rightarrow Y = \mu + \sigma Z\ ;\ Y > 0 \Rightarrow Z > -\mu/\sigma\right]$$

We have for every standard normal distribution, Z, and for every u, $Pr\left[Z > -u\right] = Pr\left[Z < u\right] = \mathbf{\Phi}\left(u\right)$. Here, ϕ and $\mathbf{\Phi}$ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E\left[Z|\,Z > -u\right] &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[\int_{-u}^{\infty} t\phi\left(t\right)dt\right] \\ &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[-\phi\left(t\right)|_{-u}^{\infty}\right] = \frac{\phi\left(u\right)}{\mathbf{\Phi}\left(u\right)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E\left[Y|\,Y > 0\right] &= \mu + \sigma E\left[Z|\,Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma\phi\left(\mu/\sigma\right)}{\mathbf{\Phi}\left(\mu/\sigma\right)} \end{aligned}$$

Setting, $\psi\left(u\right) = u + \phi\left(u\right)/\Phi\left(u\right)$,

$$E\left[Y|\,Y > 0\right] = \sigma\psi\left(\mu/\sigma\right)$$

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right)\middle|\left(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T\eta_T\right) > 0\right]$$

$$= W_T\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\left[\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]$$

$$= \left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)W_T\psi\left(\xi W_T\right)\ ,\ \xi W_T = \frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}$$

In the next to last period, $T-1$, the Bellman equation is,

$$V_{T-1}\left(P_{T-2}, X_{T-2}, W_{T-1}\right) = \min_{\{S_{T-1}\}} E_{T-1}\left[\max\left\{\left(P_{T-1}-P_{T-2}\right), 0\right\} W_{T-1} + V_T\left(P_{T-1}, X_{T-1}, W_T\right)\right]$$

$$\begin{aligned} = & \min_{\{S_{T-1}\}} E_{T-1}\left[\max\left\{\left(\theta W_{T-1}S_{T-1} + W_{T-1}\varepsilon_{T-1} + \gamma\rho W_{T-1}X_{T-2} + \gamma W_{T-1}\eta_{T-1}\right), 0\right\}\right. \\ & + \left. V_T\left(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1} + \gamma\rho X_{T-2} + \gamma\eta_{T-1}, \rho X_{T-2} + \eta_{T-1}, W_{T-1} - S_{T-1}\right)\right] \end{aligned}$$

$$\begin{aligned} = \min_{\{S_{T-1}\}} & \left\{ W_{T-1}\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\left[\frac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]\right. \\ & \left. + \left\{W_{T-1} - S_{T-1}\right\}\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\left[\frac{\theta\left(W_{T-1} - S_{T-1}\right) + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta(W_{T-1} - S_{T-1}) + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta(W_{T-1} - S_{T-1}) + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}\right]\right\} \end{aligned}$$

$$= \min_{\{S_{T-1}\}}\left[W_{T-1}\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\psi\left(\xi_1 S_{T-1}\right) + \left(W_{T-1} - S_{T-1}\right)\left(\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)\psi\left\{\xi_1\left(W_{T-1} - S_{T-1}\right)\right\}\right]$$

Here, $\xi_1 S_{T-1} = \dfrac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}}$ Also, let $\alpha = \gamma\rho X_{T-2}$, $\beta = \sqrt{\gamma^2\sigma_\eta^2 + \sigma_\varepsilon^2}$

$$\begin{aligned} = \min_{\{S_{T-1}\}} & \left\{\left[\theta W_{T-1}S_{T-1} + \alpha W_{T-1} + \beta W_{T-1}\frac{\phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}\right]\right. \\ & \left. + \left[\theta\left(W_{T-1} - S_{T-1}\right)^2 + \alpha\left(W_{T-1} - S_{T-1}\right) + \beta\left(W_{T-1} - S_{T-1}\right)\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}\right]\right\} \end{aligned}$$

This is a convex function and taking First Order Conditions give,

$$\begin{aligned} & \theta W_{T-1} + \beta W_{T-1}\left\{\frac{\theta}{\beta}\left[-\frac{\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)\phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)} - \left\{\frac{\phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1}+\alpha}{\beta}\right)}\right\}^2\right]\right\} \\ & -2\theta\left(W_{T-1} - S_{T-1}\right) - \alpha + \beta\left\{-\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}\right. \\ & \left. + \frac{\theta\left(W_{T-1} - S_{T-1}\right)}{\beta}\left[\frac{\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)} + \left\{\frac{\phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}{\Phi\left(\frac{\theta(W_{T-1}-S_{T-1})+\alpha}{\beta}\right)}\right\}^2\right]\right\} \quad = \quad 0 \end{aligned}$$

□

11.7 Proof of Proposition 6

Proof. Consider,

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\widetilde{P}_T\left(1+\theta W_T+\gamma X_T\right)-P_{T-1}\right),0\right\}W_T\right]$$

$$V_T\left(P_{T-1}, X_{T-1}, W_T\right) = E_T\left[\max\left\{\left(\widetilde{P}_{T-1}e^{B_T}\left[1+\theta W_T+\gamma\left(\rho X_{T-1}+\eta_T\right)\right]-P_{T-1}\right),0\right\}W_T\right]$$

$$= E_T\left[\max\left\{\widetilde{P}_{T-1}W_Te^{B_T}+\theta W_T^2\widetilde{P}_{T-1}e^{B_T}+\gamma\rho X_{T-1}\widetilde{P}_{T-1}W_Te^{B_T}+\gamma\widetilde{P}_{T-1}W_Te^{B_T}\eta_T-W_TP_{T-1},0\right\}\right]$$

$$\begin{aligned} = \quad & E_T\left[\left(\widetilde{P}_{T-1}W_Te^{B_T}+\theta W_T^2\widetilde{P}_{T-1}e^{B_T}+\gamma\rho X_{T-1}\widetilde{P}_{T-1}W_Te^{B_T}+\gamma\widetilde{P}_{T-1}W_Te^{B_T}\eta_T-W_TP_{T-1}\right)\right| \\ & \left.\left(\widetilde{P}_{T-1}W_Te^{B_T}+\theta W_T^2\widetilde{P}_{T-1}e^{B_T}+\gamma\rho X_{T-1}\widetilde{P}_{T-1}W_Te^{B_T}+\gamma\widetilde{P}_{T-1}W_Te^{B_T}\eta_T-W_TP_{T-1}\right)>0\right]\end{aligned}$$

$$\{\because\ E\left[max\left(X,c\right)\right] = E\left[X\left|X>c\right.\right]Pr\left[X>c\right]+E\left[c\left|X\leq c\right.\right]Pr\left[X\leq c\right]\ \}$$

This is of the form, $E\left[Y_2\right|Y_2>0]$ where,
$Y_2 = \left(\widetilde{P}_{T-1}W_Te^{B_T}+\theta W_T^2\widetilde{P}_{T-1}e^{B_T}+\gamma\rho X_{T-1}\widetilde{P}_{T-1}W_Te^{B_T}+\gamma\widetilde{P}_{T-1}W_Te^{B_T}\eta_T-W_TP_{T-1}\right)$. We simplify using some notational shortcuts,

$$E\left[\left(ae^X+be^X+ce^X+de^XY_1+k\right)\right|\left(ae^X+be^X+ce^X+de^XY_1+k\right)>0\Big]$$

$$X\sim N\left(\mu_X,\sigma_X^2\right);Y_1\sim N\left(0,\sigma_{Y_1}^2\right);X\text{ and }Y_1\text{ are independent. Also, }a,b,c,d>0,k<0$$

$$\equiv \qquad E\left[\left(e^X\left\{a+b+c+dY_1\right\}+k\right)\right|\left(e^X\left\{a+b+c+dY_1\right\}+k\right)>0\Big]$$

$$\equiv \qquad E\left[\left(e^XY+k\right)\right|\left(e^XY+k\right)>0\Big]$$

$$X\sim N\left(\mu_X,\sigma_X^2\right);Y\sim N\left(\mu_Y,\sigma_Y^2\right);X\text{ and }Y\text{ are independent. Also, }k<0$$

Consider,

$$E\left[\left(e^XY+k\right)\right|\left(e^XY+k\right)>0\Big] \quad = \quad E\left[k\left|\left(e^XY+k\right)>0\right.\right]+E\left[\left(e^XY\right)\left|\left(e^XY+k\right)>0\right.\right]$$

$$= \quad k + E\left[\left(Ye^X\right)\middle|\left(Ye^X + k\right) > 0\right]$$

$$= \quad k + \int\int ye^x f\left(ye^x \middle| \{ye^x + k\} > 0\right) dx dy$$

Here, $f(w)$ is the probability density function for w,

$$= \quad k + \int\int ye^x \frac{f\left(ye^x; \{ye^x + k\} > 0\right)}{f\left(\{ye^x + k\} > 0\right)} dx dy$$

$$\text{[We note that, } ye^x > -k > 0 \Rightarrow y > 0\text{]}$$

$$= \quad k + \int\int ye^x \frac{f(y) f\left(e^x; \{ye^x + k\} > 0\right)}{f\left(\{ye^x + k\} > 0\right)} dx dy$$

$$= \quad k + \int y \left[\int \frac{e^x f\left(e^x; \left\{e^x > -\frac{k}{y}\right\}\right)}{f\left(e^x > -\frac{k}{y}\right)} dx\right] f(y)\, dy$$

$$= \quad k + \int y \left[\int e^x f\left(e^x \middle| \left\{e^x > -\frac{k}{y}\right\}\right) dx\right] f(y)\, dy$$

$$= \quad k + \int_0^{(y<-k)} y \left[\int e^x f\left(e^x \left|\{e^x > 1\}\right.\right) dx\right] f(y)\, dy + \int_{(y>-k)}^{\infty} y \left[\int e^x f\left(e^x \left|\{e^x < 1\}\right.\right) dx\right] f(y)\, dy$$

$$= \quad k + \int_0^{(-k)} y\left[E\left(W \middle| W > c\right)\right] f(y)\, dy + \int_{(-k)}^{\infty} y\left[E\left(W \middle| W < c\right)\right] f(y)\, dy \quad ;\text{ here, } W = e^X \text{ and } c = 1$$

Simplifying the inner expectations,

$$E\left(W \middle| W > c\right) = \frac{1}{P\left(e^X > c\right)} \int_c^{\infty} w \frac{1}{w\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{ln(w)-\mu_X}{\sigma_X}\right]^2} dw$$

Put $t = ln(w)$, we have, $dw = e^t dt$

$$E\left(W \middle| W > c\right) = \frac{1}{P\left(X > ln(c)\right)} \int_{ln(c)}^{\infty} \frac{e^t}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu_X}{\sigma_X}\right)^2} dt$$

$$t - \frac{1}{2}\left(\frac{t-\mu_X}{\sigma_X}\right)^2 = -\frac{1}{2\sigma_X^2}\left(t - \left(\mu_X + \sigma_X^2\right)\right)^2 + \mu_X + \frac{\sigma_X^2}{2}$$

$$E\left(W|\,W>c\right)=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(\mu_X+\sigma_X Z>ln\left(c\right)\right)}\int_{ln(c)}^{\infty}\frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{t-\left(\mu_X+\sigma_X^2\right)}{\sigma_X}\right]^2}dt \quad ;Z\sim N\left(0,1\right)$$

Put $s=\left[\frac{t-\left(\mu_X+\sigma_X^2\right)}{\sigma_X}\right]$ and $b=\left[\frac{ln(c)-\left(\mu_X+\sigma_X^2\right)}{\sigma_X}\right]$ we have, $ds=\frac{dt}{\sigma_X}$

$$E\left(W|\,W>c\right)=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(Z>\frac{ln(c)-\mu_X}{\sigma_X}\right)}\int_{b}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}ds$$

$$=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(Z<\frac{-ln(c)+\mu_X}{\sigma_X}\right)}\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}ds-\int_{-\infty}^{b}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}ds\right]$$

$$=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(Z<\frac{-ln(c)+\mu_X}{\sigma_X}\right)}\left[1-\Phi\left(b\right)\right] \quad ;\Phi \text{ is the standard normal CDF}$$

$$=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{\Phi\left(\frac{-ln(c)+\mu_X}{\sigma_X}\right)}\left[\Phi\left(-b\right)\right]$$

Similarly for the other case,

$$E\left(W|\,W<c\right)=\frac{1}{P\left(e^X<c\right)}\int_{0}^{c}w\frac{1}{w\sigma_X\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{ln(w)-\mu_X}{\sigma_X}\right]^2}dw$$

Put $t=ln\left(w\right)$, we have, $dw=e^t dt$

$$E\left(W|\,W<c\right)=\frac{1}{P\left(X<ln\left(c\right)\right)}\int_{-\infty}^{ln(c)}\frac{e^t}{\sigma_X\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\mu_X}{\sigma_X}\right)^2}dt$$

$$t-\frac{1}{2}\left(\frac{t-\mu_X}{\sigma_X}\right)^2=-\frac{1}{2\sigma_X^2}\left(t-\left(\mu_X+\sigma_X^2\right)\right)^2+\mu_X+\frac{\sigma_X^2}{2}$$

$$E\left(W|\,W<c\right)=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(\mu_X+\sigma_X Z<ln\left(c\right)\right)}\int_{-\infty}^{ln(c)}\frac{1}{\sigma_X\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{t-\left(\mu_X+\sigma_X^2\right)}{\sigma_X}\right]^2}dt \quad ;Z\sim N\left(0,1\right)$$

Put $s=\left[\frac{t-\left(\mu_X+\sigma_X^2\right)}{\sigma_X}\right]$ and $b=\left[\frac{ln(c)-\left(\mu_X+\sigma_X^2\right)}{\sigma_X}\right]$ we have, $ds=\frac{dt}{\sigma_X}$

$$E\left(W|\,W<c\right)=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(Z<\frac{ln(c)-\mu_X}{\sigma_X}\right)}\int_{-\infty}^{b}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}ds$$

$$=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{P\left(Z<\frac{ln(c)-\mu_X}{\sigma_X}\right)}\left[\Phi\left(b\right)\right] \quad ;\Phi \text{ is the standard normal CDF}$$

$$=\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{\Phi\left(\frac{ln(c)-\mu_X}{\sigma_X}\right)}\left[\Phi\left(b\right)\right]$$

Using the results for the inner expectations,

$$E\left[\left(e^{X}Y+k\right)\middle|\left(e^{X}Y+k\right)>0\right]=k+\int_{0}^{(-k)}y\left[\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{\Phi\left(\frac{-ln(c)+\mu_X}{\sigma_X}\right)}\left[\Phi\left(-b\right)\right]\right]f\left(y\right)dy+\int_{(-k)}^{\infty}y\left[\frac{e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}}{\Phi\left(\frac{ln(c)-\mu_X}{\sigma_X}\right)}\left[\Phi\left(b\right)\right]\right]f\left(y\right)dy$$

$$=k+e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}\left[\int_{0}^{(-k)}y\left\{\frac{\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}f\left(y\right)dy+\int_{(-k)}^{\infty}y\left\{\frac{\Phi\left(-\left[\frac{\mu_X+\sigma_X^2}{\sigma_X}\right]\right)}{\Phi\left(-\left[\frac{\mu_X}{\sigma_X}\right]\right)}\right\}f\left(y\right)dy\right]$$

$$=k+e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}\left[\int_{0}^{(-k)}y\left\{\frac{\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}f\left(y\right)dy+\int_{(-k)}^{\infty}y\left\{\frac{1-\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{1-\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}f\left(y\right)dy\right]$$

$$\begin{aligned}=\quad & k+e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}\left[\left\{\frac{\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}\int_{-\frac{\mu_Y}{\sigma_Y}}^{-\left(\frac{k+\mu_Y}{\sigma_Y}\right)}\left(\mu_Y+\sigma_Y z\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\right.\\ & \left.+\left\{\frac{1-\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{1-\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}\int_{-\left(\frac{k+\mu_Y}{\sigma_Y}\right)}^{\infty}\left(\mu_Y+\sigma_Y z\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz\right]\quad ;Z\sim N\left(0,1\right)\end{aligned}$$

$$\begin{aligned}=\quad & k+e^{\left(\mu_X+\frac{1}{2}\sigma_X^2\right)}\left[\left\{\frac{\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}\left\{\mu_Y\left[\Phi\left(-\left[\frac{k+\mu_Y}{\sigma_Y}\right]\right)-\Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)\right]-\frac{\sigma_Y}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{k+\mu_Y}{\sigma_Y}\right)^2}-e^{-\frac{1}{2}\left(\frac{\mu_Y}{\sigma_Y}\right)^2}\right]\right\}\right.\\ & \left.+\left\{\frac{1-\Phi\left(\frac{\mu_X+\sigma_X^2}{\sigma_X}\right)}{1-\Phi\left(\frac{\mu_X}{\sigma_X}\right)}\right\}\left\{\mu_Y\left[1-\Phi\left(-\left[\frac{k+\mu_Y}{\sigma_Y}\right]\right)\right]+\frac{\sigma_Y}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left(\frac{k+\mu_Y}{\sigma_Y}\right)^2}\right]\right\}\right]\end{aligned}$$

□

11.8 Proof of Proposition 7

Proof. Consider,

$$V_T\left(P_{T-1},O_{T-1},W_T\right)=\min_{\{S_T\}}E_T\left[\max\left\{\left(P_T-P_{T-1}\right),0\right\}S_T\right]$$

$$V_T\left(P_{T-1},O_{T-1},W_T\right)=E_T\left[\max\left\{\left(\alpha P_{T-1}+\theta W_T P_{T-1}-\gamma\left(O_T-W_T\right)P_{T-1}+\varepsilon_T\right),0\right\}W_T\right]$$

$$=E_T\left[\max\left\{\left(\alpha P_{T-1}W_T+\theta W_T^2 P_{T-1}+\gamma W_T^2 P_{T-1}-\gamma\rho O_{T-1}W_T P_{T-1}-\gamma W_T P_{T-1}\eta_T+W_T\varepsilon_T\right),0\right\}\right]$$

Setting $\beta = \theta + \gamma$,

$$V_T\left(P_{T-1}, O_{T-1}, W_T\right) = E_T\left[\left(\alpha P_{T-1}W_T + \beta W_T^2 P_{T-1} - \gamma\rho O_{T-1}W_T P_{T-1} - \gamma W_T P_{T-1}\eta_T + W_T\varepsilon_T\right)\right| \left(\alpha P_{T-1}W_T + \beta W_T^2 P_{T-1} - \gamma\rho O_{T-1}W_T P_{T-1} - \gamma W_T P_{T-1}\eta_T + W_T\varepsilon_T\right) > 0\Big]$$

$$\{\because\ E\left[max\left(X, c\right)\right] = E\left[X\left|X > c\right.\right]Pr\left[X > c\right] + E\left[c\left|X \le c\right.\right]Pr\left[X \le c\right]\ \}$$

This is of the form, $E\left[Y|\, Y > 0\right]$ where,

$Y = \left(\alpha P_{T-1}W_T + \beta W_T^2 P_{T-1} - \gamma\rho O_{T-1}W_T P_{T-1} - \gamma W_T P_{T-1}\eta_T + W_T\varepsilon_T\right)$. We then need to calculate,

$$E_T\left[\left\{\alpha P_{T-1}W_T + \beta W_T^2 P_{T-1} - \gamma\rho O_{T-1}W_T P_{T-1} + W_T\left(\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2 + \sigma_\varepsilon^2}\right)Z\right\}\right| Z > \left(-\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma\rho O_{T-1}P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2 + \sigma_\varepsilon^2}}\right)\Bigg] \quad , where\ Z \sim N\left(0,1\right)$$

$$[\because\ X \sim N(\mu_X, \sigma_X^2)\ ;\ Y \sim N(\mu_Y, \sigma_Y^2)\ ;\ U = X + Y \Rightarrow U \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)]$$

$$\Bigg[\because\ Y \sim N\left(\alpha P_{T-1}W_T + \beta W_T^2 P_{T-1} - \gamma\rho O_{T-1}W_T P_{T-1} - \gamma W_T P_{T-1}\eta_T + W_T\varepsilon_T\right) \equiv Y \sim N\left(\mu, \sigma^2\right) \Rightarrow Y = \mu + \sigma Z\ ;\ Y > 0 \Rightarrow Z > -\mu/\sigma\Bigg]$$

We have for every standard normal distribution, Z, and for every u, $Pr\left[Z > -u\right] = Pr\left[Z < u\right] = \mathbf{\Phi}\left(u\right)$. Here, ϕ and $\mathbf{\Phi}$ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E\left[Z|\, Z > -u\right] &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[\int_{-u}^{\infty} t\phi\left(t\right)dt\right] \\ &= \frac{1}{\mathbf{\Phi}\left(u\right)}\left[-\phi\left(t\right)|_{-u}^{\infty}\right] = \frac{\phi\left(u\right)}{\mathbf{\Phi}\left(u\right)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E\left[Y|\, Y > 0\right] &= \mu + \sigma E\left[Z|\, Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma\phi\left(\mu/\sigma\right)}{\mathbf{\Phi}\left(\mu/\sigma\right)} \end{aligned}$$

Setting, $\psi\left(u\right) = u + \phi\left(u\right)/\Phi\left(u\right)$,

$$E\left[Y|\, Y > 0\right] = \sigma\psi\left(\mu/\sigma\right)$$

$$V_T(P_{T-1}, O_{T-1}, W_T) = W_T\left(\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)\left[\left(\frac{\alpha P_{T-1}+\beta W_T P_{T-1}-\gamma\rho O_{T-1}P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right) + \frac{\phi\left(\frac{\alpha P_{T-1}+\beta W_T P_{T-1}-\gamma\rho O_{T-1}P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\alpha P_{T-1}+\beta W_T P_{T-1}-\gamma\rho O_{T-1}P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}\right]$$

$$= \left(\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2+\sigma_\varepsilon^2}\right) W_T\psi(\xi W_T)\ ,\ \xi W_T = \left(\frac{\alpha P_{T-1}+\beta W_T P_{T-1}-\gamma\rho O_{T-1}P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)$$

In the next to last period, $T-1$, the Bellman equation is,

$$V_{T-1}(P_{T-2}, O_{T-2}, W_{T-1}) = \min_{\{S_{T-1}\}} E_{T-1}\left[\max\{(P_{T-1}-P_{T-2}),0\} S_{T-1} + V_T(P_{T-1}, O_{T-1}, W_T)\right]$$

$$= \min_{\{S_{T-1}\}} E_{T-1}\Big[\max\left\{\left(\alpha P_{T-2}S_{T-1}+\beta S_{T-1}^2 P_{T-2}-\gamma\rho O_{T-2}S_{T-1}P_{T-2}-\gamma S_{T-1}P_{T-2}\eta_{T-1}+S_{T-1}\varepsilon_{T-1}\right),0\right\}$$
$$+ \quad V_T\left((\alpha+1)P_{T-2}+\beta S_{T-1}P_{T-2}-\gamma\rho O_{T-2}P_{T-2}-\gamma P_{T-2}\eta_{T-1}+\varepsilon_{T-1}, \rho O_{T-2}+\eta_{T-1}, W_{T-1}-S_{T-1}\right)\Big]$$

$$= \min_{\{S_{T-1}\}} \quad E_{T-1}\Bigg\{ S_{T-1}\left(\sqrt{\gamma^2 P_{T-2}^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)$$

$$\left[\left(\frac{\alpha P_{T-2}+\beta S_{T-1}P_{T-2}-\gamma\rho O_{T-2}P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right) + \frac{\phi\left(\frac{\alpha P_{T-2}+\beta S_{T-1}P_{T-2}-\gamma\rho O_{T-2}P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\alpha P_{T-2}+\beta S_{T-1}P_{T-2}-\gamma\rho O_{T-2}P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}\right]$$

$$+(W_{T-1}-S_{T-1})\left(\sqrt{\gamma^2\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}^2\sigma_\eta^2+\sigma_\varepsilon^2}\right)$$

$$\left[\left(\frac{\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}\left\{\alpha+\beta(W_{T-1}-S_{T-1})-\gamma\rho^2 O_{T-2}-\gamma\rho\eta_{T-1}\right\}}{\sqrt{\gamma^2\left\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\right\}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)\right.$$

$$\left.+\frac{\phi\left(\frac{\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\}\{\alpha+\beta(W_{T-1}-S_{T-1})-\gamma\rho^2 O_{T-2}-\gamma\rho\eta_{T-1}\}}{\sqrt{\gamma^2\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\}\{\alpha+\beta(W_{T-1}-S_{T-1})-\gamma\rho^2 O_{T-2}-\gamma\rho\eta_{T-1}\}}{\sqrt{\gamma^2\{P_{T-2}(\alpha+1+\beta S_{T-1}-\gamma\rho O_{T-2}-\gamma\eta_{T-1})+\varepsilon_{T-1}\}^2\sigma_\eta^2+\sigma_\varepsilon^2}}\right)}\right]\Bigg\}$$

□

12 Summary of Related Papers

Building on the foundation laid by (Bertsimas & Lo 1998), another popular way to decompose trading costs is into temporary and permanent impact (See Almgren & Chriss 2001; Almgren 2003; and Almgren, Thum, Hauptmann & Li 2005). While the theory behind this approach is extremely elegant and considers both linear and nonlinear functions of the variables for estimating the impact, a practical way to compute it requires measuring the price a certain interval after the order. This interval is ambiguous and could lead to lower accuracy while using this measure.

More recent extensions include: (Huberman & Stanzl 2005) minimize the mean and variance of the costs of trading for the case of market orders only and derive explicit formulas for the optimal trading strategies, showing that risk-averse liquidity traders (someone that wishes to trade a fixed number of shares within a certain time horizon) reduce their order sizes over time and execute a higher fraction of their total trading volume in early periods when price volatility or liquidity increases. (Forsyth, Kennedy, Tse & Windcliff 2012) argue that quadratic variation can be regarded as a reasonable risk measure (rather than variance) and derive the Hamilton Jacobi Bellman (HJB) Partial Differential Equations (PDE) and provide numerical methods to solve for both the optimal strategies and the efficient frontier with arbitrary constraints on the strategy, assuming that the asset price dynamic follows either Geometric Brownian Motion (GBM) or Arithmetic Brownian Motion (ABM).

(Almgren & Lorenz 2007) derive optimal strategies where the execution accelerates when the price moves in the trader's favor, and slows when the price moves adversely; (Kissell & Malamut 2006) term such adaptive strategies "aggressive-in-the-money"; A "passive-in-the-money" strategy would react oppositely. (Schied, Schöneborn & Tehranchi 2010) consider the problem faced by an investor who must liquidate a given basket of assets over a finite time horizon. They assume that the investor's utility has constant absolute risk aversion (CARA[5]) and that the asset prices are given by a very general continuous-time, multi-asset price impact model and show that the investor does no worse if he narrows his search to deterministic strategies.

(Schied & Schöneborn 2009) use a stochastic control approach[6], building upon the continuous time model of (Almgren 2003), and show that the value function and optimal control satisfy certain nonlinear parabolic partial differential equations that can be solved numerically. (Kato 2014) develops a mathematical model of optimal execution, by formulating it as a stochastic control problem in the continuous time domain. The continuity of the value function and the semigroup property (Bellman principle) are investigated with the findings that the value function is continuous in each parameter except for the time origin, where the right-continuity at $t = 0$, depends on the market impact function. The semigroup property suggests that the value

[5]CARA has exponential utility of the form $u(c) = 1 - e^{-\alpha c}$, so that the absolute risk aversion, $A(c) = -\frac{u''(c)}{u'(c)} = \alpha$, a constant. Wikipedia Link on Risk Aversion.

[6](Wikipedia Link on Stochastic Control: Stochastic control or stochastic optimal control is a subfield of control theory that deals with the existence of uncertainty either in observations or in the noise that drives the evolution of the system.)

function is characterized as a viscosity solution of the corresponding HJB equation, which is a nonlinear second order PDE.

(Gatheral & Schied 2011) find a closed-form solution for the optimal trade execution strategy in the Almgren-Chriss framework assuming the underlying unaffected stock price (stock price before the impact or before the transaction occurs) process is a GBM; (Schied 2013) investigates the robustness of this strategy with respect to misspecification of the law of the underlying unaffected stock price process. (Guo & Zervos 2015) study the optimal execution problem in the context of a continuous time model with multiplicative price impact, involving singular control rather than absolutely continuous control: this setting does not restrict stock transactions to be realized at a rate over time; instead, it allows for block sales of stock. In classical control problems (Shreve 1988), the cumulative displacement of the state, caused by control, is the integral of the control process (or some function of it), and so is absolutely continuous. In impulse control, this cumulative displacement has jumps, between which it is either constant or absolutely continuous. Bounded variation control (defined to include any stochastic control problem in which one restricts the cumulative displacement of the state caused by control to be of bounded variation on finite time intervals) admits both these possibilities and also the possibility that the displacement of the state caused by the optimal control is singularly continuous, at least with positive probability over some interval of time.

Building on empirical evidence that instantaneous market impact is a strongly concave function of the volume (Lillo, Farmer & Mantegna 2003), well approximated by a power law function, at least for trading rates that are not too high; (Curato, Gatheral & Lillo 2017) find that the discretized cost function exhibits a rugged landscape, with many local minima separated by peaks. (Brunnermeier & Pedersen 2005; Carlin, Lobo & Viswanathan 2007) are extensions to situations with several competing traders, wherein if one trader is forced to liquidate his holdings, other traders also sell, creating downward price pressure, and buy back the assets later at a lower price.

(Huberman & Stanzl 2004) provide theoretical arguments showing that in the absence of quasi-arbitrage (availability of a sequence of round-trip trades that generate infinite expected profits with an infinite Sharpe ratio, that is infinite expected profits per unit of risk), permanent price-impact functions must be linear; though empirical investigations suggest that the shape of the limit order book (LOB) can be more complex (Hopman 2007).

In contrast to many studies, where the dynamics of the asset price process is taken as a given fundamental, (Obizhaeva & Wang 2013) proposed a market impact model that derives its dynamics from an underlying model of a LOB. In this model, the ask part of the LOB consists of a uniform distribution of shares offered at prices higher than the current best ask price. When the large trader is not active, the mid price of the LOB fluctuates according to the actions of noise traders, and the bid-ask spread remains constant. A buy market order of the large trader, however, consumes a block of shares located immediately to the right of the best ask and thus increase the ask price by a linear proportion of the size of the order. In addition, the LOB will recover from the impact of the buy order, i.e., it will show a certain resilience. The resulting price impact will neither be instantaneous nor entirely permanent but will decay on an exponential scale. Bid-ask spread and market depth capture the static aspects of liquidity, related to the shape of the limit order book, which determines how much the current price moves in response to a trade. Resilience reflects the dynamic aspect of liquidity, related to how the future limit-order book evolves in response to the current trade.

(Alfonsi, Fruth & Schied 2010) extend this by allowing for a general shape of the LOB defined via a given density function, which can accommodate empirically observed LOB shapes and obtain a nonlinear price impact of market orders. They also allow for dynamic updating of trading strategies and intermediate sell orders. The resilience of the LOB after a large market order is modeled as having an exponential recovery of the number of limit orders, i.e., of the volume of the LOB, or the exponential recovery of the bid-ask spread. (Predoiu, Shaikhet & Shreve 2011) derive optimal strategies, (under a general shape of the LOB), that are a mixture of lump purchases and continuous purchases with the rate of purchase set to match the order book resilience.

(Fruth, Schöneborn & Urusov 2014) analyze optimal strategies, for a risk neutral investor, when liquidity varies deterministically (liquidity is time dependent; depth and resilience can be independently time-dependent in contrast to the LOB model of Obizhaeva & Wang 2013) and find that in the case of extreme changes in liquidity, it can even be optimal to completely refrain from trading in periods of low liquidity. Price manipulations under such a scenario are ruled out by considering a time dependent spread, which widens when liquidity is low and a trader buys in large quantity hoping to sell it later and make a profit without depressing market prices in periods of high liquidity. Empirical studies based on the LOB model are (Biais, Hillion & Spatt 1995; Potters & Bouchaud 2003; Bouchaud, Gefen, Potters & Wyart 2004; and Weber & Rosenow 2005).

A related strand of literature looks at models of the LOB from the perspective of dealers seeking to submit optimal strategies (maximize the utility of total terminal wealth) of bid and ask orders. (Ho & Stoll 1981) analyze the optimal prices for a monopolistic dealer in a single stock when faced with a stochastic demand to trade, modeled by a continuous time Poisson jump process, and facing return uncertainty, modeled by diffusion processes. (Ho and Stoll 1980), consider the problem of dealers under competition (each dealer's pricing strategy depends not only on his own current and expected inventory position and his other characteristics, but also on the current and expected inventory and other characteristics of the competitor) and show that the bid and ask prices are shown to be related to the reservation (or indifference) prices of the agents.

(Avellaneda & Stoikov 2008) combine the utility framework with the microstructure of actual limit order books, as described in the econo-physics literature, to infer reasonable arrival rates of buy and sell orders; (Du, Zhu & Zhao 2016) extend the price dynamics to follow a GBM in which the drift part is updated by

Bayesian learning in the beginning of the transaction day to capture the trader's estimate of other traders' target sizes and directions. (Cont, Stoikov & Talreja 2010) describe a stylized model for the dynamics of a limit order book (which serves as a comprehensive introduction to limit order books), where the order flow is described by independent Poisson processes and estimate the model parameters from high-frequency order book time-series data from the Tokyo Stock Exchange.

(Cont, Kukanov & Stoikov 2014) study the price impact of order book events - limit orders, market orders, and cancellations - using the NYSE Trades and Quotes data for fifty randomly selected stocks. They show a linear relation between order flow imbalance, (OFI, defined as the imbalance between supply and demand at the best bid and ask prices) and price changes, with a slope inversely proportional to the market depth. The OFI explains price changes better than the trade imbalance, (defined as the difference between volumes of buyer and seller-initiated trades), during a given interval, and is a more general measure of supply/demand imbalance because it adequately includes the effect of trade imbalance.

(Cont & Kukanov 2017) focus on the order placement problem, which is to choose an order type - market or limit order - and which trading venue(s) to submit it to, when there are multiple alternatives. They derive an optimal split between market and limit orders for a single exchange and extend the results to the general case of order placement on multiple trading venues. A numerical algorithm for solving the order placement problem in a general case is provided using a robust modification of the Robbins-Monro stochastic approximation technique (Robbins & Monro 1951; Nemirovski, Juditsky, Lan & Shapiro 2009).

(Guo, de Larrard & Ruan 2017) derive optimal placement strategies for both static and dynamic cases (in the static case, as opposed to the dynamic case, a strategy is completely decided before execution takes place, that is at $t = 0$, and is unchanged over the entire order internal), under a correlated random walk model, with mean-reversion for the best ask/bid price (the spread between the best bid price and the best ask price is always one tick and the best ask price increases or decreases one tick at each time step; also the change in the ask price is a Markov chain, with probability that makes prices mean reverting). In the static case, the optimal strategy involves only the market order, the best bid, and the second best bid; whereas for the dynamic case it depends on the remaining trading time, the market momentum, and the price mean-reversion factor.

(Gabaix, Gopikrishnan & Stanley 2006) present a theory in which spikes in trading volume and returns, and hence stock market volatility, are created by a combination of news and the trades of large investors. Spikes in market activity can imply that the empirical moments might be infinite; requiring returns, trading volume, price impact and the size of large institutional investors to follow power law distributions, which is supported by plenty of empirical evidence. The model explains the power law distribution of price impact and reconciles the power law of returns and trading volume, while deriving the optimal behavior of institutional investors.

While our work focuses on separating impact and timing in the (Bertsimas & Lo 1998) framework; a natural and interesting continuation would be to extend this separation to models of the limit order book.

Models of market impact and the design of better trading strategies are becoming an integral part of the present trend at automation and the increasing use of algorithms. (Jain 2005) assembles the dates of announcement and actual introduction of electronic trading by the leading exchange of 120 countries to examine the long term and medium term impact of automation. He finds that automation of trading on a stock exchange has a long-term impact on listed firms' cost of equity. Estimates from the dividend growth model, as well as the international CAPM, suggest a significant decline in expected returns after the introduction of electronic trading in the world's equity markets, especially in the developing nations and confirms the finding from previous studies that electronic trading improves a stock's liquidity and reduces investors' trading costs. (Hendershott, Jones & Menkveld 2011) perform an empirical study on New York Stock Exchange stocks and find that algorithmic trading and liquidity are positively related.

It is worth noting a contrasting result from an earlier study. (Venkataraman 2001) compares securities on the New York Stock Exchange (NYSE) (a floor-based trading structure with human intermediaries, specialists, and floor brokers) and the Paris Bourse (automated limit-order trading structure). He finds that execution costs might be higher on automated venues even after controlling for differences in adverse selection, relative tick size, and economic attributes. A trade occurs when an aggressive trader submits a market order and demands liquidity, hence the rules on a venue are designed to attract demanders of liquidity and nudge liquidity providers to display their orders. Displaying limit orders involves risks. First, the counter-parties could be better informed, and liquidity providers could get picked off. Hence, they would like the trading system to allow them to trade selectively with counter-parties of their choice. Second, they risk being front-run by other traders with an increase in the market impact of their orders. Hence, large traders want to hide their orders and expose them only to traders who are most likely to trade with them. This means fully automated exchanges, which anecdotally seems to be the way ahead, need to take special care to formulate rules, to help liquidity providers better control the risks of order exposure.

What this also means is that, the design of better strategies and models is crucial to survive and thrive in this continuing trend at automation. Our paper aims to fill the gap in existing models of trading costs, which are theoretically elegant but are not readily applicable to real life trading situations, since they do not allow participants to gauge how they are performing in comparison to the other participants with whom they are competing for liquidity. Our models have a strong theoretical foundation but they can be applied to actual trading situations due to the insights they provide to participants. In addition, our numerical framework can be be used to obtain optimal execution schedules under any law of motion of prices.